Notes on motives in finite characteristic

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To Yuri Manin on the occasion of 70-th birthday, with admiration.

Introduction and an example

These notes grew from an attempt to interpret a formula of Drinfeld (see [3]) enumerating the absolutely irreducible local systems of rank 2 on algebraic curves over finite fields, obtained as a corollary of the Langlands correspondence for $GL(2)$ in the functional field case, and of the trace formula.

Let C be a smooth projective geometrically connected curve defined over a finite field $\mathbb{F}_q$, with a base point $v \in C(\mathbb{F}_q)$. The geometric fundamental group $\pi_1^{geom}(C,v) := \pi_1(C \times_{\mathtt{Spec}\,\mathbb{F}_q} \mathtt{Spec}\,\overline{\mathbb{F}}_q, v)$ is a profinite group on which the Galois group $\widehat{\mathbb{Z}} = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ (with the canonical generator $\mathrm{Fr} := \mathrm{Fr}_q$) acts. In what follows we will omit the base point from the notation.

Theorem 1. (Drinfeld) *Under the above assumptions, for any integer $n \geq 1$ and any prime $l \neq char(\mathbb{F}_q)$ the set of fixed points*

$$X_n^{(l)} := \left(\mathrm{IrrRep}\,(\pi_1(C \times_{\mathtt{Spec}\,\mathbb{F}_q} \mathtt{Spec}\,\overline{\mathbb{F}}_q) \to GL(2, \overline{\mathbb{Q}}_l))/conjugation\right)^{\mathrm{Fr}^n}$$

is finite. Here $\mathrm{IrrRep}\,(\dots)$ *denotes the set of conjugacy classes of irreducible continuous 2-dimensional representations of $\pi_1^{geom}(C)$ defined over finite extensions of $\mathbb{Q}_l$. Moreover, there exists a finite collection $(\lambda_i) \in \overline{\mathbb{Q}}^\times$ of algebraic integers, and signs $(\epsilon_i) \in \{-1,+1\}$ depending only on C, such that for any n, l one has an equality*

$$\# X_n^{(l)} = \sum_i \epsilon_i \lambda_i^n \ .$$

From the explicit formula which one can extract from [3] one can see that numbers λ_i are q-Weil algebraic integers whose norm for any embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ belongs to $q^{\frac{1}{2}\mathbb{Z}_{\geq 0}}$. Therefore, the number of elements of $X_n^{(l)}$, $n = 1, 2, \dots$

looks like the number of $\mathbb{F}_{q^n}$-points on some variety over $\mathbb{F}_q$. The largest exponent is q^{4g-3}, which indicates that this variety has dimension $4g-3$. A natural guess is that it is closely related to the moduli space of stable bundles of rank 2 over C. At least the dimensions coincide, and Weil numbers which appear are essentially the same, they are products of the eigenvalues of Frobenius acting on the motive defined by the first cohomology of C.

The Langlands correspondence identifies $X_n^{(l)}$ with the set of $\overline{\mathbb{Q}}_l$-valued unramified cuspidal automorphic forms for the adelic group $GL(2, \mathbb{A}_{\mathbb{F}_q(C)})$. These forms are eigenvectors of a collection of commuting matrices (Hecke operators) with integer coefficients. Therefore, for a given $n \geq 1$ one can identify[1] all sets $X_n^{(l)}$ for various primes l with one set X_n endowed with an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, extending the obvious actions of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$ on $X_n^{(l)}$.

These days the Langlands correspondence in the functional field case is established for all the groups $GL(N)$ by L. Lafforgue. To my knowledge, almost no attempts were made to extend Drinfeld's calculation to the case of higher rank, or even to the $GL(2)$ case with non-trivial ramification.

It is convenient to take the inductive limit $X_\infty := \varinjlim X_n, \;\; X_{n_1} \hookrightarrow X_{n_1 n_2}$ which is an infinite countable set endowed with an action of the product[2]

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \;\; .$$

The individual set X_n can be reconstructed from this datum as the set of fixed points of $\mathrm{Fr}^n \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

In spite of the numerical evidence, it would be too naive to expect a natural identification of X_∞ with the set of $\overline{\mathbb{F}}_q$-points of an algebraic variety defined over $\mathbb{F}_q$, as there is no obvious mechanism producing a non-trivial $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the latter.

The main question addressed here is

Question 2. Does there exist some alternative way to construct the set X_∞ with the commuting action of two Galois groups?

In the present notes I will offer three different hypothetical constructions. The first construction comes from the analogy between the Frobenius acting on $\pi_1^{geom}(C)$ and an element of the mapping class group acting on the fundamental group of a closed oriented surface, the second one is almost tautological and arises from the contemplation on the shape of explicit formulas for Hecke operators (see an example in Section 0.1), the third one is based on an analogy with lattice models in statistical physics.

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[1] It is expected that all representations from $X_n^{(l)}$ are motivic, i.e. they arise from projectors with coefficients in $\overline{\mathbb{Q}}$ acting on l-adic cohomology of certain projective varieties defined over the field of rational functions $\mathbb{F}_{q^n}(C)$.

[2] One can replace $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by its quotient $\mathrm{Gal}(\mathbb{Q}^{q-Weil}/\mathbb{Q})$ where $\mathbb{Q}^{q-Weil} \subset \overline{\mathbb{Q}}$ is CM-field generated by all q-Weil numbers.

I propose several conjectures, which should be better considered as guesses in the first and in the third part, as there is almost no experimental evidence in their favor. In a sense, the first and the third part should be regarded as science fiction, but even if the appropriate conjectures are wrong (as I strongly suspect), there should be some grains of truth in them.

On the contrary, I feel quite confident that the conjectures made in the second part are essentially true, the output is a higher-dimensional generalization of the Langlands correspondence in the functional field case. At the end of the second part I will show how to make a step in the arithmetic direction, extending the formulas to the case of an arbitrary local field.

In the fourth part I will describe briefly a similarity between a modification of the category of motives based on non-commutative geometry, and two other categories introduced in the second and the third part. Also I will make a link between the proposal based on polynomial dynamics and the one based on lattice models.

Finally, I apologize to the reader that the formulas in Sections 0.1 and 1.3 are given without explanations, this is the result of my poor knowledge of the representation theory. The formulas were polished with the help of computer.

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0.1 An explicit example

Here we will show explicit formulas for the tower $(X_n)_{n\geq 1}$ in the simplest truly non-trivial case. Consider the affine curve $C = \mathbb{P}^1_{\mathbb{F}_q} \setminus \{0, 1, t, \infty\}$ for a given element $t \in \mathbb{F}_q \setminus \{0, 1\}$. We are interested in motivic local $SL(2)$-systems on C with tame non-trivial unipotent monodromies around all punctures $\{0, 1, t, \infty\}$.

A lengthy calculation lead to the following explicit formulas[3] for the Hecke operators for cuspidal representations. In what follows we assume $\operatorname{char} \mathbb{F}_q \neq 2$. The Hecke operators act on the spaces of functions on certain double coset space for the adelic group, which can be identified with the set of equivalence classes of vector bundles of rank 2 over $\mathbb{P}^1_{\mathbb{F}_q}$ together with a choice of

[3] I was informed by V. Drinfeld that a similar calculation for the case of $SL(2)$ local systems on $\mathbb{P}^1_{\mathbb{F}_q} \setminus \{4 \text{ points}\}$, with tame non-trivial *semisimple* monodromy around punctures was performed few years ago by Teruji Thomas.

one-dimensional subspaces of fibers at $\{0,1,t,\infty\}$. This double coset space is infinite, but the eigenfunctions of Hecke operators corresponding to cuspidal representations have finite support which one can bound a priori.

For any $x \in \mathbb{F}_q$ the Hecke operator T_x (on cuspidal forms) can be written as an integral $q \times q$ matrix whose rows and columns are labelled by elements of $\mathbb{F}_q$, (i.e. $T_x \in \mathrm{Mat}(\mathbb{F}_q \times \mathbb{F}_q; \mathbb{Z})$). Coefficients of T_x are given by the formula

$$(T_x)_{yz} := 2 - \#\{w \in \mathbb{F}_q |\, w^2 = f_t(x,y,z)\} + (\text{correction term})$$

where $f_t(x,y,z)$ is the following universal polynomial with integral coefficients:

$$f_t(x,y,z) := (xy + yz + zx - t)^2 + 4xyz(1 + t - (x+y+z)) \ .$$

The correction term is equal to

$$-\begin{cases} q+1 & x = y \in \{0,1,t\} \\ 1 & x = y \notin \{0,1,t\} \\ 0 & x \neq y \end{cases} + \begin{cases} q & \text{if } x \notin \{0,1,t\} \ \text{ and } \begin{cases} y = \dfrac{t}{x}, & z = 0 \\ y = \dfrac{t-x}{1-x}, & z = 1 \\ y = \dfrac{t(1-x)}{t-x}, & z = t \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

Operators $(T_x)_{x \in \mathbb{F}_q}$ satisfy the following properties:

1. $[T_{x_1}, T_{x_2}] = 0$,
2. $\sum_{x \in \mathbb{F}_q} T_x = \mathbf{1} = id_{\mathbb{Z}^{\mathbb{F}_q}}$,
3. $T_x^2 = \mathbf{1}$ for $x \in \{0,1,t\}$, moreover $\{\mathbf{1}, T_0, T_1, T_t\}$ form a group under the multiplication, isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$,
4. for any $x \notin \{0,1,t\}$ the spectrum of T_x is real and belongs to $[-2\sqrt{q}, +2\sqrt{q}]$, any element of $\mathtt{Spec}(T_x)$ can be written as $\lambda + \overline{\lambda}$ where $|\lambda| = \sqrt{q}$ is a q-Weil number,
5. for any $\xi = \lambda + \overline{\lambda} \in \mathtt{Spec}(T_x)$ and any integer $n \geq 1$ the spectrum of the matrix $T_x^{(n)}$ corresponding to $x \in \mathbb{F}_q \subset \mathbb{F}_{q^n}$ (if we pass to the extension $\mathbb{F}_{q^n} \supset \mathbb{F}_q$) contains the element $\xi^{(n)} := \lambda^n + \overline{\lambda}^n$,
6. the vector space generated by $\{T_x\}_{x \in \mathbb{F}_q}$ is closed under the product, the multiplication table is

$$T_x \cdot T_y = \sum_{z \in \mathbb{F}_q} c_{xyz} T_z \ \text{ where } c_{xyz} = (T_x)_{yz} \ .$$

Typically (for "generic" t, x) the characteristic polynomial of T_x splits into the product of 4 irreducible polynomials of almost the same degree. The splitting is not surprising, as we have a group[4] of order 4 commuting with

[4] This is the group of automorphisms of $\mathbb{P}^1 \setminus \{4 \text{ points}\}$ for the generic cross-ratio.

all operators T_x (see property 3). Computer experiments indicate that the Galois groups of these polynomials (considered as permutation groups) tend to be rather large, typically the full symmetric groups if q is prime, and the corresponding number fields have huge factors in the prime decomposition of the discriminant.

Notice that in the theory of automorphic forms one usually deals with infinitely many commuting Hecke operators corresponding to all places of the global field, i.e. to closed points of C (in other words, to orbits of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\overline{\mathbb{F}}_q)$ acting on $C(\overline{\mathbb{F}}_q)$). Here we are writing formulas only for the points defined over $\mathbb{F}_q$. The advantage of our example is that the number these operators coincides with the size of Hecke matrices, hence one can try to write formulas for structure constants, which by luck turn out to coincide with the matrix coefficients of matrices T_x (property 6).

1 First proposal: algebraic dynamics

As was mentioned before, it is hard to imagine a mechanism for a non-trivial action of the absolute Galois group of $\mathbb{Q}$ on the set of points of a variety over a finite field. One can try to exchange the roles of fields $\mathbb{Q}$ and $\mathbb{F}_q$. The first proposal is the following one:

Conjecture 3. *For a tower $(X_n)_{n\geq 1}$ arising from automorphic forms (or from motivic local systems on curves), as defined in the Introduction, there exists a variety X defined over $\mathbb{Q}$ and a map $F : X \to X$ such that there is a family of bijections*

$$X_n \simeq (X(\overline{\mathbb{Q}}))^{F^n},\ \ n \geq 1$$

covariant with respect to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{Z}/n\mathbb{Z}$ *actions, and with respect to inclusions $X_{n_1} \subset X_{n_1 n_2}$ for integers $n_1, n_2 \geq 1$.*

1.1 The case of $GL(1)$

Geometric class field theory gives a description of the sets $(X_n)_{n\geq 1}$ in terms of the Jacobian of C:

$$X_n = (\mathrm{Jac}_C(\mathbb{F}_{q^n}))^\vee(\overline{\mathbb{Q}}) = \mathrm{Hom}(\mathrm{Jac}_C(\mathbb{F}_{q^n}), \overline{\mathbb{Q}}^\times)\ .$$

The number of elements of this set is equal to

$$\#\,\mathrm{Jac}_C(\mathbb{F}_{q^n})\ = \det(\mathrm{Fr}^n_{H^1(C)} - \mathbf{1})$$

where $\mathrm{Fr}_{H^1(C)}$ is the Frobenius operator acting on, say, l-adic first cohomology group of C.

One can propose a blatantly non-canonical candidate for the corresponding dynamical system (X, F). Namely, let us choose a semisimple $(2g\times 2g)$ matrix

$A = (A_{i,j})_{1\leq i,j\leq 2g}$ (where g is the genus of C) with coefficients in $\mathbb{Z}$, whose characteristic polynomial is equal to the characteristic polynomial of $\mathrm{Fr}_{H^1(C)}$. Define $X/\mathbb{Q}$ to be the standard $2g$-dimensional torus $\mathbb{G}_m^{2g} = \mathrm{Hom}(\mathbb{Z}^{2g}, \mathbb{G}_m)$, and the map F to be the dual to the map $A : \mathbb{Z}^{2g} \to \mathbb{Z}^{2g}$:

$$F(z_1, \dots, z_{2g}) = (\prod_i z_i^{A_{i,1}}, \dots, \prod_i z_i^{A_{i,2g}}) \ .$$

Moreover, one can choose A in such a way that

$$F^*\omega = q\omega \quad \text{ where } \omega = \sum_{i=1}^{g} \frac{dz_i}{z_i} \wedge \frac{dz_{g+i}}{z_{g+i}} \ .$$

On the set of fixed points of F^n act simultaneously $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (via the cyclotomic quotient) and $\mathbb{Z}/n\mathbb{Z}$ (by powers of F). Nothing contradicts to the existence of an equivariant isomorphism between two towers of finite sets.

1.2 Moduli of local systems on surfaces

One can interpret the scheme $\mathbb{G}_m^{2g}$ as the moduli space of rank 1 local systems on a oriented closed topological surface S of genus g, the form ω is the natural symplectic form on this moduli space.

In general, for any $N \geq 1$, one can make an analogy between the action of Frobenius Fr on the the set of l-adic irreducible representations of $\pi_1^{geom}(C)$ of rank N, and the action of the isotopy class of a homeomorphism $\varphi : S \to S$ on the set of irreducible complex representations of $\pi_1(S)$ of the same rank. Sets of representations are similar to each other, as it is known that the maximal quotient of $\pi_1^{geom}(C)$ coprime to q is isomorphic to the analogous quotient of the profinite completion $\widehat{\pi}_1(S)$ of $\pi_1(S)$. Also, if we assume that there are only finitely many fixed points of φ acting on

$$\mathrm{IrrRep}(\pi_1(S) \to GL(2, \mathbb{C}))/\text{conjugation}$$

then the sets

$$X^{(l)} := \left(\mathrm{IrrRep}(\widehat{\pi}_1(C \times_{\mathtt{Spec}\, \mathbb{F}_q} \mathtt{Spec}\, \overline{\mathbb{F}}_q) \to GL(2, \overline{\mathbb{Q}}_l))/\text{conjugation}\right)^{\varphi}$$

do not depend on the prime l for l large enough.

All this leads to the following conjecture (which is formulated a bit sloppily), a strengthening of Conjecture 1:

Conjecture 4. *For any smooth compact geometrically connected curve $C/\mathbb{F}_q$ of genus $g \geq 2$ there exists an endomorphism Φ_C of the tensor category of finite-dimensional complex local systems on S such that*

- *Φ_C is algebraic and defined over $\mathbb{Q}$, in the sense that it acts on the moduli stack of irreducible local systems of any given rank $N \geq 1$ by a rational map defined over $\mathbb{Q}$,*

- *Φ_C multiplies the natural symplectic form on the moduli space of irreducible local systems of rank N by the constant q,*
- *for every $n, N \geq 1$ there exists an identification of the set of isomorphism classes of irreducible motivic local systems of rank N on $C \times_{\mathtt{Spec}\mathbb{F}_q} \mathtt{Spec}\,\overline{\mathbb{F}}_q$ invariant under Fr^n, with the set of isomorphism classes of $\overline{\mathbb{Q}}$-local systems of rank N on S invariant under Φ_C^n, compatible with the relevant Galois symmetries and tensor constructions.*

One can not expect that Φ_C comes from an actual endomorphism φ of the fundamental group $\pi_1(S)$, as it is known that for $g \geq 2$ any such φ is necessarily an automorphism. That is a rationale for replacing a putative endomorphism of $\pi_1(S)$ by a more esoteric endomorphism of the tensor category of its finite-dimensional representations.

Example: $SL(2)$-local systems on sphere with 3 punctures

A generic $SL(2,\mathbb{C})$-local system on $\mathbb{C}P^1 \setminus \{0,1,\infty\}$ is uniquely determined by 3 traces of monodromies around punctures. A similar statement holds for l-adic local systems with tame monodromy in the case of finite characteristic. Motivic local systems correspond to the case when all the eigenvalues of the monodromies around punctures are roots of unity, i.e. when the traces of monodormies are twice cosines of rational angles. This leads to the following prediction:

$$X = \mathbb{A}^3, \;\; F(x_1, x_2, x_3) = (T_q(x_1), T_q(x_2), T_q(x_3))$$

where $T_q \in \mathbb{Z}[x]$ is the q-th Chebyshev polynomial,

$$T_q(\lambda + \lambda^{-1}) = \lambda^q + \lambda^{-q} \;\; .$$

In this case the identifications between the fixed points of F and motivic local systems on $\mathbb{P}^1_{\mathbb{F}_q} \setminus \{0,1,\infty\}$ exist, and can be extracted form the construction of these local systems (called hypergeometric) as summands in certain direct images of abelian local systems (analogous the classical integral formulas for hypergeometric functions). The identification is ambiguous, it depends on a choice of a group embedding $\overline{\mathbb{F}}_q^\times \hookrightarrow \mathbb{Q}/\mathbb{Z}$.

1.3 Equivariant bundles and Ruelle-type zeta-functions

The analogy with an element of the mapping class group acting on surface S suggest the following addition to the Conjecture 1. Let us fix the curve $C/\mathbb{F}_q$ and the rank $N \geq 1$ of local systems under the consideration. For a given point $x \in C(\mathbb{F}_q)$ we have a sequence of Hecke operators $T_x^{(n)}$ associated with curves $C \times_{\mathtt{Spec}\,\mathbb{F}_q} \mathtt{Spec}\,\mathbb{F}_{q^n}$. The spectrum of $T_x^{(n)}$ is a $\overline{\mathbb{Q}}$-valued function on X_n, i.e. according to Conjecture 1, a function on the set of fixed points of F^n. We expect that the collection of these functions for $n = 1, 2\ldots$ comes from a F-equivariant vector bundle on X.

Conjecture 5. *Using the notations of Conjecture 1, for given $x \in C(\mathbb{F}_q)$ there exists a pair $(\mathcal{E}, g)$ where $\mathcal{E}$ is a vector bundle on X of rank N together with an isomorphism $g : F^*\mathcal{E} \to \mathcal{E}$ (defined over $\mathbb{Q}$), such that the eigenvalue of $T_x^{(n)}$ at the point of its spectrum corresponding to $z \in X_n$ coincides with*

$$\mathrm{Trace}(\mathcal{E}_z = \mathcal{E}_{F^n(z)} \to \cdots \to \mathcal{E}_{F(z)} \to \mathcal{E}_z)$$

where arrows are isomorphisms of fibers of $\mathcal{E}$ coming from g.

In particular, one can ask for an explicit formula for the F-equivariant bundle $\mathcal{E}$ in the case of $SL(2)$-local systems on the sphere with 3 punctures where we have an explicit candidate for (X, F).

In the limiting most simple non-abelian case when the monodromy is unipotent around 2 punctures, and arbitrary semisimple around the third puncture, one can make the above question completely explicit:

Question 6. For a given $x \in \mathbb{F}_q \setminus \{0, 1\}$, does there exist a rational function $R = R_x$ on $\mathbb{C}P^1$ with values in $q^{1/2}SL(2, \mathbb{C}) \subset \mathrm{Mat}(2 \times 2, \mathbb{C})$ which has no singularities on the set

$$\left(\cup_{n \geq 1} \{z \in \mathbb{C} | z^{q^n - 1} = 1\} \right) \setminus \{1\}$$

such that for any $n \geq 1$ two sets of complex numbers (with multiplicities):

$$X_n := \left\{ \sum_{y \in \mathbb{F}_{q^n} \setminus \{0,1,x\}} \chi\left(\frac{y(1-xy)}{1-y} \right) \mid \chi : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times,\ \chi \neq 1 \right\}$$

where χ runs through all non-trivial multiplicative characters of $\mathbb{F}_{q^n}$, and

$$X'_n := \left\{ \mathrm{Trace}\left(R(z)R(z^q)\dots R(z^{q^{n-1}}) \right) \mid z^{q^n - 1} = 1,\ z \neq 1 \right\}$$

coincide?

Elements of the set X_n are real numbers of the form $\lambda + \overline{\lambda}$ where $\lambda \in \overline{\mathbb{Q}}$ is a q-Weil number with $|\lambda| = q^{1/2}$. Therefore it is natural to expect that $R(z)$ belongs to $q^{1/2}SU(2)$ if $|z| = 1$.

The Galois symmetry does not forbid for the function R (as a rational function with values in (2×2)-matrices) to be defined over $\mathbb{Q}$, after the conjugation by a constant matrix. Moreover, the existence of such a function over $\mathbb{Q}$ leads to certain choice of generators of the multiplicative groups $\left(\mathbb{F}_{q^n}^\times\right)$ for all $n \geq 1$, well-defined modulo the action of Frobenius Fr_q (the Galois group $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \mathbb{Z}/n\mathbb{Z}$), as in a sense we identify roots of unity in $\mathbb{C}$ and multiplicative characters of $\mathbb{F}_{q^n}$. In particular, there will be a *canonical* irreducible polynomial over $\mathbb{F}_q$ of degree n for every $n \geq 1$. This is something almost too good to be true.

Reminder: Trace formula and Ruelle-type zeta-function

Let X be now a smooth *proper* variety (say, over $\mathbb{C}$), endowed with a map $F : X \to X$, and $\mathcal{E}$ be a vector bundle on X together with a morphism (not necessarily invertible) $g : F^*\mathcal{E} \to \mathcal{E}$. Let us assume that for any $n \geq 1$ all fixed points z of $F^n : X \to X$ are isolated and *non-degenerate*, i.e. the tangent map

$$(F^n)'_z : T_zX \to T_zX$$

has no nonzero invariant vectors (in other words, all eigenvalues of $(F^n)'_z$ are not equal to 1). Then one has the following identity (Atiyah-Bott fixed point formula):

$$\sum_{v\in X: F^n(z)=z} \frac{\mathrm{Trace}(\mathcal{E}_z = \mathcal{E}_{F^n(z)} \to \cdots \to \mathcal{E}_z)}{\det(1-(F^n)'_z)} =$$
$$= \mathrm{Trace}((g_* \circ F^*)^n : H^\bullet(X,\mathcal{E}) \to H^\bullet(X,\mathcal{E}))$$

The trace in the r.h.s. is understood in the super sense, as the alternating sum of the ordinary traces in individual cohomology spaces.

If one wants to eliminate the determinant factor in the denominator in the l.h.s., one should replace $\mathcal{E}$ by the superbundle $\mathcal{E} \otimes \wedge^\bullet(T^*_X)$.

The trace formula implies that the series in t

$$\exp\left(-\sum_{n\geq 1} \frac{t^n}{n} \sum_{z\in X: F^n(z)=z} \frac{\mathrm{Trace}(\mathcal{E}_z = \mathcal{E}_{F^n(z)} \to \cdots \to \mathcal{E}_z)}{\det(1-(F^n)'_z)}\right)$$

is the Taylor expansion of a rational function in t. It seems that in many cases for *non-compact* varieties X a weaker form of rationality holds as well, when no equivariant compactification can be found. Namely, the above series (called the Ruelle zeta-function in general, not necessarily algebraic case) admits a meromorphic continuation to $\mathbb{C}$; also the zeta-function in the version without the denominator is often rational in the non-compact case.

Rationality conjecture for motivic local systems

In the case hypothetically corresponding to motivic local systems on curves (in the setting of Conjecture 3), one can make a natural a priori guess about the denominator in the l.h.s. of the trace formula. Namely, for a fixed point z of the map F^n corresponding to a fixed point $[\rho]$ in the space of representations of $\pi_1(C \times_{\mathtt{Spec}\,\mathbb{F}_q} \mathtt{Spec}\,\overline{\mathbb{F}}_q)$ in $GL(N, \overline{\mathbb{Q}}_l)$, we expect that the vector space T_zX together with the automorphism $(F^n)'_z$ should be isomorphic (after the change of scalars) to

$$H^1(C \times_{\mathtt{Spec}\,\mathbb{F}_q} \mathtt{Spec}\,\overline{\mathbb{F}}_q, \mathrm{End}(\rho)) = \mathrm{Ext}^1(\rho,\rho)$$

endowed with the Frobenius operator.

Eigenvalues of Fr^n in this case have norm $q^{n/2}$ by the Weil conjecture, hence not equal to 1, and the denominator in the Ruelle zeta-function does not vanish (meaning that the fixed points are non-degenerate).

In our basic example from Section 0.1 one can propose an explicit formula for the denominator term. Define (in notation from Section 0.1) for given $t \in \mathbb{F}_q \setminus \{0,1\}$ a matrix $T_{tan} \in \mathrm{Mat}(\mathbb{F}_q \times \mathbb{F}_q, \mathbb{Q})$ by the formula

$$T_{tan} := -\frac{1}{q} \sum_{x \in \mathbb{F}_q} (T_x)^2 + (q - 3 - 1/q) \cdot id_{\mathbb{Q}^{\mathbb{F}_q}} \quad .$$

This matrix satisfies the same properties as Hecke operators[5]. Namely, all eigenvalues of T_{tan} belong to $[-2\sqrt{q}, +2\sqrt{q}]$, any element of $\mathtt{Spec}(T_{tan})$ can be written as $\lambda + \overline{\lambda}$ where $|\lambda| = \sqrt{q}$ is a q-Weil number, and for any $\xi = \lambda + \overline{\lambda} \in \mathtt{Spec}(T_x)$ and any integer $n \geq 1$ the spectrum of the matrix $T_{tan}^{(n)}$ corresponding to $x \in \mathbb{F}_q \subset \mathbb{F}_{q^n}$ (if we pass to the extension $\mathbb{F}_{q^n} \supset \mathbb{F}_q$) contains the element $\xi^{(n)} := \lambda^n + \overline{\lambda}^n$.

We expect that the eigenvalue of T_{tan} at the point of the spectrum corresponding to motivic local system ρ is equal to the trace of Frobenius in a two-dimensional submotive of the motive $H^1(C, \mathrm{End}(\rho))$, corresponding to the deformations of ρ preserving the unipotency of the monodromy around punctures.

Notice that any motivic local system ρ on C can be endowed with a non-degenerate skew-symmetric pairing with values in the Tate motive. This explains the main term of the formula:

- the sum of squares of Hecke operators means that we are using the trace formula for Frobenius in the cohomology of C with coefficients in the tensor square of ρ,
- the factor $1/q$ comes from the Tate twist,
- the minus sign comes from the odd (first) cohomology.

The candidate for the denominator term in the putative Ruelle zeta-function is the following operator commuting with the Hecke operators (we write the formula only for the first iteration, $n = 1$), considered as a function on the spectrum:

$$D := (q + 1 - T_{tan})^{-1} \quad .$$

The reason is that the eigenvalue of D at the eigenvector corresponding to motivic local system ρ is equal to

$$\frac{1}{(1-\lambda)(1-\overline{\lambda})} = \frac{1}{1 + q - \xi}$$

where $\lambda, \overline{\lambda}$ are Weil numbers, eigenvalues of Frobenius in $H^1(C, \mathrm{End}(\rho))$ satisfying equations

[5]The only difference is that eigenvalues of operators T_x are algebraic integers while eigenvalues of T_{tan} are algebraic integers divided by q.

$$\lambda + \overline{\lambda} = \xi, \ \lambda + \overline{\lambda} = q \ .$$

The l.h.s. of the putative trace formula for the equivariant vector bundle $\mathcal{E}_{x_1} \otimes \dots \otimes \mathcal{E}_{x_k}$ (here $\mathcal{E}_x$ is the F-equivariant vector bundle corresponding to point $x \in C(\mathbb{F}_q)$, see Conjecture 3), is given (for the n-th iteration) by the formula

$$\mathrm{Trace}\left(T^{(n)}_{x_1} \dots T^{(n)}_{x_k} D^{(n)}\right) \ .$$

It looks that in order to achieve the rationality of the putative Ruelle zeta-function one has to add by hand certain contributions corresponding to "missing fixed points". For example, for any $x \in \mathbb{F}_q \setminus \{0, 1, t\}$ one has

$$\mathrm{Trace}(T_x D) = \frac{q}{(q-1)^2}$$

and the corresponding zeta-function

$$\exp\left(-\sum_{n\geq 1} \frac{t^n}{n} \frac{q^n}{(q^n-1)^2}\right) = \prod_{m\geq 1} (1 - q^{-m} t)^m \in \mathbb{Q}[[t]]$$

is meromorphic but not rational. The above zeta-function looks like the contribution of just one[6] fixed point z_0 on an algebraic dynamical system $z \mapsto F(z)$ on a two-dimensional variety, with the spectrum of $(F')_{z_0}$ equal to (q, q), and the spectrum of the map on the fiber $\mathcal{E}_{z_0} \to \mathcal{E}_{z_0}$ equal to $(q, 0)$. Here is the precise conjecture coming from computer experiments:

Conjecture 7. *For any* $x_1, \dots, x_k \in \mathbb{F}_q \setminus \{0, 1, t\}, \ k \geq 1$ *the series*

$$\exp\left(-\sum_{n\geq 1} \frac{t^n}{n} \left\{\mathrm{Trace}\left(T^{(n)}_{x_1} \dots T^{(n)}_{x_k} D^{(n)}\right) + \mathrm{Corr}(n, k)\right\}\right)$$

where

$$\mathrm{Corr(n,k)} := -\frac{\left(-1-q^n\right)^k}{\left(1-q^{-n}\right)\left(1-q^{2n}\right)} \ ,$$

is a rational function.

The rational function in the above conjecture should be an L-function of a motive over $\mathbb{F}_q$, all its zeroes and poles should be q-Weil numbers.

Finally, if one considers Ruelle zeta-functions *without* the denominator term, then rationality is elementary, as will become clear in the next section.

[6] Maybe the complete interpretation should be a bit more complicated as one can check numerically that $Trace(D) = \frac{q^2(q-2)}{(q-1)^2(q+1)}$.

2 Second proposal: formalism of motivic function spaces and higher-dimensional Langlands correspondence

The origin of this section is property 6 (the multiplication table) of Hecke operators in our example from Section 0.1.

2.1 Motivic functions and the tensor category $\mathcal{C}_{\mathbf{k}}$

Let S be a noetherian scheme.

Definition 8. The commutative ring $\mathrm{Fun}^{poor}(S)$ of poor man's motivic functions[7] on S is the quotient of the free abelian group generated by equivalence classes of schemes of finite type over S, modulo relations

$$[X \to S] = [Z \to S] + [(X \setminus Z) \to S]$$

where Z is a closed subscheme of X over S. The multiplication on $\mathrm{Fun}^{poor}(S)$ is given by the fibered product over S.

In the case when S is the spectrum of a field $\mathbf{k}$, we obtain the standard definition[8] of the Grothendieck ring of varieties over $\mathbf{k}$. Any motivic function on S gives a function on the set of points of S with values in the Grothendieck rings corresponding to the residue fields.

For a given field $\mathbf{k}$ let us consider the following additive category $\mathcal{C}_{\mathbf{k}}$. Its objects are schemes of finite type over $\mathbf{k}$, the abelian groups of homomorphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}_{\mathbf{k}}}(X, Y) := \mathrm{Fun}^{poor}(X \times Y) \ .$$

The composition of two morphisms represented by schemes is given by the fibered product as below:

$$[B \to Y \times Z] \circ [A \to X \times Y] := [A \times_Y B \to X \times Z]$$

and extended by additivity to all motivic functions. The identity morphism id_X is given by the diagonal embedding $X \hookrightarrow X \times X$.

One can start from the beginning from constructible sets over $\mathbf{k}$ instead of schemes. The category of constructible sets over $\mathbf{k}$ is a full subcategory of $\mathcal{C}_{\mathbf{k}}$, the morphism in $\mathcal{C}_{\mathbf{k}}$ corresponding to a constructible map $f : X \to Y$ is given by $[X \stackrel{(id_X, f)}{\longrightarrow} X \times Y]$, the graph of f.

Finite sums (and products) in $\mathcal{C}_{\mathbf{k}}$ are given by the disjoint union.

We endow category $\mathcal{C}_{\mathbf{k}}$ with the following tensor structure on objects:

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[7] The name was suggested by V. Drinfeld.

[8] Usually one extends the Grothendieck ring of varieties by inverting the class $[A^1_{\mathbf{k}}]$ of the affine line, which is the geometric counterpart of the inversion of the Lefschetz motive $L = H_2(\mathbb{P}^1_k)$ in the construction of Grothendieck pure motives. Here also we can do the same thing.

$$X \otimes Y := X \times Y$$

and by a similar formula on morphisms. The unit object $\mathbf{1}_{\mathcal{C}_\mathbf{k}}$ is the point $Spec(\mathbf{k})$. The category $\mathcal{C}_\mathbf{k}$ is rigid, i.e. for every object X there exists another object $X^\vee$ together with morphisms $\delta_X : X \otimes X^\vee \to \mathbf{1},\ \epsilon_X : \mathbf{1} \to X^\vee \otimes X$ such that both compositions:

$$X \stackrel{id_X \otimes \epsilon_X}{\longrightarrow} X \otimes X^\vee \otimes X \stackrel{\delta_X \otimes id_X}{\longrightarrow} X,\ X^\vee \stackrel{\epsilon_X \otimes id_{X^\vee}}{\longrightarrow} X^\vee \otimes X \otimes X^\vee \stackrel{id_{X^\vee} \otimes \delta_X}{\longrightarrow} X^\vee$$

are identity morphisms. The dual object $X^\vee$ in $\mathcal{C}_\mathbf{k}$ coincides with X, the duality morphisms $\delta_X,\ \epsilon_X$ are given by the diagonal embedding $X \hookrightarrow X^2$.

As in any tensor category, the ring $\mathrm{End}_{\mathcal{C}_\mathbf{k}}(\mathbf{1}_{\mathcal{C}_\mathbf{k}})$ is commutative, and the whole category is linear over this ring, which is nothing but the Grothendieck ring of varieties over $\mathbf{k}$.

Fiber functors for finite fields

If $\mathbf{k} = \mathbb{F}_q$ is a finite field then there is an infinite chain $(\phi_n)_{n \geq 1}$ of tensor functors from $\mathcal{C}_\mathbf{k}$ to the category of finite-dimensional vector spaces over $\mathbb{Q}$. It is defined on objects by the formula

$$\phi_n(X) := \mathbb{Q}^{X(\mathbb{F}_{q^n})} \ .$$

The operator corresponding by ϕ_n to a morphism $[A \to X \times Y]$ has the following matrix coefficient with indices $(x,y) \in X(\mathbb{F}_{q^n}) \times Y(\mathbb{F}_{q^n})$:

$$\#\{a \in A(\mathbb{F}_{q^n}) \,|\, a \mapsto (x,y)\} \in \mathbb{Z}_{\geq 0} \subset \mathbb{Q}$$

The functor ϕ_n is not canonically defined for $n \geq 2$, the ambiguity is the cyclic group $\mathbb{Z}/n\mathbb{Z} = \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \subset \mathrm{Aut}(\phi_n)$.

Extensions and variants

The abelian group $\mathrm{Fun}^{poor}(S)$ of poor man's motivic functions can (and probably should) be replaced by the K_0 group of the triangulated category $Mot_{S,\mathbb{Q}}$ of "constructible motivic sheaves" (with coefficients in $\mathbb{Q}$) on S. Although the latter category is not yet rigorously defined, one can envision a reasonable candidate for the elementary description of $K_0(Mot_{S,\mathbb{Q}})$. This group should be generated by equivalence classes of families of Grothendieck motives (with coefficients in $\mathbb{Q}$) over closed subschemes of S, modulo a suitable equivalence relation. Moreover, group $K_0(Mot_{S,\mathbb{Q}})$ should be filtered by the dimension of support, the associated graded group should be canonically isomorphic to the direct sum over all points $x \in S$ of K_0 groups of categories of pure motives (with coefficients in $\mathbb{Q}$) over the residue fields[9].

[9] I do not know how to fill all the details in the above sketch.

Similarly, one can extend the coefficients of the motives from $\mathbb{Q}$ to any field of zero characteristic. This change will affect the group K_0 and give a different algebra of motivic functions.

Finally, one can add formally images of projectors to the category $\mathcal{C}_{\mathbf{k}}$.

Question 9. Are there interesting non-trivial projectors in $\mathcal{C}_{\mathbf{k}}$?

I do not know at the moment any example of an object in the Karoubi closure of $\mathcal{C}_{\mathbf{k}}$ which is not isomorphic to a scheme. Still, there are interesting non-trivial isomorphisms between objects of $\mathcal{C}_{\mathbf{k}}$, for example the following version of the Radon transform.

Example: motivic Radon transform

Let $X = \mathbb{P}(V)$ and $Y = \mathbb{P}(V^\vee)$ be two dual projective spaces over $\mathbf{k}$. We assume that $n := \dim V$ is at least 3.

The incidence relation gives a subvariety $Z \subset X \times Y$, which can be interpreted as a morphism in $\mathcal{C}_{\mathbf{k}}$ in two ways:

$$f_1 := [Z \hookrightarrow X \times Y] \in \mathrm{Hom}_{\mathcal{C}_{\mathbf{k}}}(X, Y)$$

$$f_2 := [Z \hookrightarrow Y \times X] \in \mathrm{Hom}_{\mathcal{C}_{\mathbf{k}}}(Y, X)$$

The composition $f_2 \circ f_1$ is equal to

$$[\mathbb{A}^{n-2}] \cdot id_X + [\mathbb{P}^{n-3}] \cdot [X \to pt \to X] \ .$$

The reason is that the scheme of hyperplanes passing through points $x_1, x_2 \in X$ is either $\mathbb{P}^{n-3}$ if $x_1 \neq x_2$, or $\mathbb{P}^{n-2}$ if $x_1 = x_2$. On the level on constructible sets one has $\mathbb{P}^{n-2} = \mathbb{P}^{n-3} \sqcup \mathbb{A}^{n-2}$.

The first term is the identity morphism multiplied by the $(n-2)$-nd power of the Tate motive, while the second term is in a sense a rank 1 operator. It can be killed after passing to the quotient of X by pt which is in fact a direct summand in $\mathcal{C}_{\mathbf{k}}$:

$$X \simeq pt \oplus (X \setminus pt) \ .$$

Here we have to choose a point $pt \in X$. Similar arguments work for Y, and as the result we obtain an isomorphism (inverting the Tate motive)

$$X \setminus pt \simeq Y \setminus pt$$

in the category $\mathcal{C}_{\mathbf{k}}$ which is not a geometric isomorphism of constructible sets.

2.2 Commutative algebras in $\mathcal{C}_{\mathbf{k}}$

By definition, a unital commutative associative algebra A in the tensor category $\mathcal{C}_{\mathbf{k}}$ is given by a scheme of finite type $X/\mathbf{k}$, and two elements

$$\mathbf{1}_A \in \mathrm{Fun}^{poor}(X),\ m_A \in \mathrm{Fun}^{poor}(X^3)$$

(the unit and the product in A) satisfying the usual constraints of unitality, commutativity and associativity.

The formula for the structure constants $c_{xyz} = (T_x)_{yz}$ of the algebra of Hecke operators in our basic example (see 0.1) is given explicitly by counting points on constructible sets depending constructibly on a point $(x, y, z) \in X^3$ where $X = \mathbb{A}^1$, for any $t \in \mathbf{k} \setminus \{0, 1\}$ (one should replace factors q by bundles with fiber $\mathbb{A}^1$). Hence we obtain a motivic function on X^3 which gives the structure of a commutative algebra on X for any $t \in \mathbf{k} \setminus \{0, 1\}$, for arbitrary field $\mathbf{k}$. A straightforward check (see Proposition 1 in Section 2.4 below for a closely related statement) shows that this algebra is associative.

Elementary examples of algebras

The first example of a commutative algebra is given by an arbitrary scheme X (or a constructible set) of finite type over $\mathbf{k}$. The multiplication tensor is given by the diagonal embedding $X \hookrightarrow X^3$, the unit is given by the identity map $X \to X$. If $\mathbf{k} = \mathbb{F}_q$ is finite then for any $n \geq 1$ the algebra $\phi_n(X)$ is the algebra of $\mathbb{Q}$-valued functions on the finite set $X(\mathbb{F}_{q^n})$, with the pointwise multiplication.

The next example corresponds to the case when X is an abelian group scheme (e.g. $\mathbb{G}_a$, $\mathbb{G}_m$, or an abelian variety). We define the multiplication tensor $m_A \in \mathrm{Fun}^{poor}(X^3)$ as the graph of the multiplication morphism $X \times X \to X$. Again, if $\mathbf{k}$ is finite then the algebra $\phi_n(X)$ is the group algebra with coefficients in $\mathbb{Q}$ of the finite abelian group $X(\mathbb{F}_{q^n})$. Its points in $\overline{\mathbb{Q}}$ are additive (resp. multiplicative) characters of $\mathbf{k}$ if $X = \mathbb{G}_a$ (resp. $X = \mathbb{G}_m$).

Also, one can see that the algebra in $\mathcal{C}_{\mathbf{k}}$ corresponding to the group scheme $\mathbb{G}_a$ is isomorphic to the direct sum of $\mathbf{1}_{\mathcal{C}_{\mathbf{k}}}$ (corresponding to the trivial additive character of $\mathbf{k}$) and another algebra A' which can be thought as parameterizing *non-trivial* additive characters of the field, with the underlying scheme $\mathbb{A}^1 \setminus \{0\}$.

Finally, one can make "quotients" of abelian group schemes by finite groups of automorphisms. For example, for $\mathbb{G}_a$ endowed with the action of the antipodal involution $x \to -x$, the formula for the product for the corresponding algebra is the sum of the following "main term"

$$[Z \hookrightarrow (\mathbb{A}^1)^3], \quad Z = \{(x, y, z) |\ x^2 + y^2 + z^2 - 2(xy + yz + zx) = 0\}$$

(the latter equation means that $\sqrt{x} + \sqrt{y} = \sqrt{z}$), and of certain correction terms. Similarly, for the antipodal involution $(x, w) \to (x, -w)$ on the elliptic curve $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ given by $w^2 = x(x-1)(x-t)$ (with (∞, ∞) serving as zero for the group law), the quotient is $\mathbb{P}^1$ endowed with the multiplication law similar to one from the example 0.1. The main term is given by the hypersurface $f_t(x, y, z) = 0$ in the notation from Section 0.1. The spectrum of the corresponding algebra is rather trivial, in comparison to our example. The difference is that in Section 0.1 we consider the two-fold cover of $(\mathbb{A}^1)^3$ ramified at the hypersurface $f_t(x, y, z) = 0$.

Categorification

One may wonder whether a commutative associative algebra A in $\mathcal{C}_{\mathbf{k}}$ (for general field $\mathbf{k}$, not necessarily finite) is in fact a materialization of the structure of a symmetric (or only braided) monoidal category on a triangulated category, i.e. whether the multiplication morphism is the class in K_0 of a bifunctor defining the monoidal structure. The category under consideration should be either the category of constructible mixed motivic sheaves on the underlying scheme of A, or some small modification of it not affecting the group K_0 (e.g. both categories could have semi-orthogonal decompositions with the same factors).

2.3 Algebras parameterizing motivic local systems

As we noticed already, the Example 0.1 can be interpreted as a commutative associative algebra in $\mathcal{C}_{\mathbf{k}}$ parameterizing in a certain sense (via the chain of functors $(\phi_n)_{n\geq 1}$) motivic local systems on a curve over $\mathbf{k} = \mathbb{F}_q$. Here we will formulate a general conjecture, which goes beyond the case of curves.

Preparations on ramification and motivic local systems

Let Y be a smooth geometrically connected projective variety over a finitely generated field $\mathbf{k}$. Let us denote by K the field of rational functions on X and by K' the field of rational functions on $Y' := Y \times_{\mathtt{Spec}\,\mathbf{k}} \mathtt{Spec}\,\overline{\mathbf{k}}$. We have an exact sequence

$$1 \to \mathrm{Gal}(\overline{K}/K') \to \mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}) \to 1$$

For a continuous homomorphism

$$\rho : \mathrm{Gal}(\overline{K}/K') \to GL(N, \overline{\mathbb{Q}}_l)$$

where $l \neq \mathrm{char}(\mathbf{k})$, which factorizes through the quotient $\pi_1^{geom}(U)$ for some open subscheme $U \subset Y'$ one can envision some notion of ramification divisor (similar to the notion of the conductor in one-dimensional case) which should be an effective divisor on Y'.

One expects that for a pure motive of rank N over K with coefficients in $\overline{\mathbb{Q}}$, the ramification divisor of the corresponding l-adic local system does not depend on prime $l \neq \mathrm{char}(\mathbf{k})$, at least for large l.

Denote by $\mathrm{IrrRep}_{Y',N,l}$ the set of conjugacy classes of irreducible representations $\rho : \mathrm{Gal}(\overline{K}/K) \to GL(N, \overline{\mathbb{Q}}_l)$ factorizing through $\pi_1^{geom}(U)$ for some open subscheme $U \subset Y'$ as above. The Galois group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts on this set.

Denote by $\mathrm{IrrRep}_{Y,N}^{mot,geom}$ the set of equivalence classes of pure motives in the sense of Grothendieck (defined using the numerical equivalence) of rank N over K, with coefficients $\overline{\mathbb{Q}}$, which are absolutely simple (i.e. remain simple after the pullback to K'), modulo the action of the Picard group of rank 1 motives over $\mathbf{k}$ with coefficients in $\overline{\mathbb{Q}}$. This set is endowed with a natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The superscript *geom* indicates that we are interested only in representations of the geometric fundamental group.

One expects that the natural map from $\mathrm{IrrRep}_{Y,N}^{mot,geom}$ to the set of fixed points $(\mathrm{IrrRep}_{Y',N,l})^{\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})}$ is a bijection. In particular, it implies that one can define the ramification divisor for an element of $\mathrm{IrrRep}_{Y,N}^{mot,geom}$. Presumably, one can give a purely geometric definition of it, without referring to l-adic representations.

Conjecture on algebras parameterizing motivic local systems

Conjecture 10. *For a smooth projective geometrically connected variety Y over a finite field $\mathbf{k} = \mathbb{F}_q$, an effective divisor D on Y, and a positive integer N, there exists a commutative associative unital algebra $A = A_{Y,D,N}$ in the category $\mathcal{C}_{\mathbf{k}}$ satisfying the following property:*

For any $n \geq 1$ the algebra $\phi_n(A)$ over $\mathbb{Q}$ is semisimple (i.e. it is a finite direct sum of number fields) and for any prime l, $(l,q) = 1$ there exists a bijection between $\mathrm{Hom}_{\mathbb{Q}-alg}(\phi_n(A), \overline{\mathbb{Q}})$ and the set of elements of $\mathrm{IrrRep}^{mot,geom}_{Y \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\,\mathbb{F}_{q^n}, N}$ for which the ramification divisor is D. Moreover, the above bijection is equivariant with the respect to the natural $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{Z}/n\mathbb{Z}$-action.

One can also try to formulate a generalization of the above conjecture, allowing not an individual variety Y but a family, i.e. a smooth projective morphism $\mathcal{Y} \to B$ to a scheme of finite type over k, with geometrically connected fibers, together with a flat family of ramification divisors. The corresponding algebra should parameterize choices of a point $b \in B(\mathbb{F}_{q^n})$ and a irreducible motivic system of given rank and a given ramification on the fiber $\mathcal{Y}_b$. This algebra should map to the algebra of functions with the pointwise product (see 2.2) associated with the base B.

In the above conjecture we did not describe how to associate a *tower* of finite sets to the algebra A, as a priori we have just a *sequence* of finite sets $X_n := \mathrm{Hom}_{\mathbb{Q}-alg}(\phi_n(A), \overline{\mathbb{Q}})$ without no obvious maps between them. This leads to the following

Question 11. Which property of an associative commutative algebra A in $\mathcal{C}_{\mathbb{F}_q}$ gives naturally a chain of embeddings

$$\mathrm{Hom}_{\mathbb{Q}-alg}(\phi_{n_1}(A), \overline{\mathbb{Q}}) \hookrightarrow \mathrm{Hom}_{\mathbb{Q}-alg}(\phi_{n_1 n_2}(A), \overline{\mathbb{Q}})$$

for all integers $n_1, n_2 \geq 1$?

It looks that this holds automatically, by a kind of trace morphism.

Arguments in favor, and extensions

First of all, there is a good reason to believe that Conjecture 5 holds for curves. Also, it would be reasonable to consider local systems with an arbitrary structure group G instead of $GL(N)$. The algebra parameterizing motivic local system on curve $Y = C$ with structure group G should be (roughly) equal to some finite open part of the moduli stack Bun_{G^L} of G^L-bundles on C, where G^L is the Langlands dual group. The multiplication should be given by the class of a motivic constructible sheaf on

$$(Bun_{G^L})^3 = Bun_{G^L} \times Bun_{G^L \times G^L}$$

which should be a geometric counterpart to the lifting of automorphic forms corresponding to the diagonal embedding

$$G^L \to G^L \times G^L \quad .$$

Presumably, the multiplication law from Example 0.1 corresponds to the lifting.

If we believe in the Conjecture 5 in the case of curves, then it is very natural to believe in it in general. The reason is that for a higher-dimensional variety Y (not necessarily compact) there exists a curve $C \subset Y$ such that $\pi_1^{geom}(Y)$ is a *quotient* of $\pi_1^{geom}(C)$. Such a curve can be e.g. a complete intersection of ample divisors, the surjectivity is a particular case of the Lefschetz theorem on hyperplane sections. Therefore, the set of equivalence classes of absolutely irreducible motivic local systems on $Y \times_{\mathtt{Spec}\,\mathbb{F}_q} \mathtt{Spec}\,\mathbb{F}_{q^n}$ should be a *subset* of the corresponding set for C for any $n \geq 1$, and invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action as well. It looks very plausible that such a collection of subsets should arise from a quotient algebra in $\mathcal{C}_{\mathbb{F}_q}$.

From the previous discussion it looks that the motivic local systems in higher-dimensional case are "less interesting", the 1-dimensional case is the richest one. Nevertheless, there is definitely a non-trivial higher-dimensional information about local systems which can not be reduced to 1-dimensional data. Namely, for any motivic local system ρ^{arith} on Y and an integer $i \geq 0$ the cohomology space

$$H^i(Y', \rho)$$

where ρ is the pullback of ρ^{arith} to Y', is a motive over the finite field $\mathbf{k} = \mathbb{F}_q$. We can calculate the trace of N-th power of Frobenius on it for a given $N \geq 1$, and get a $\overline{\mathbb{Q}}$-valued function[10] on the set

[10]Here there is a small ambiguity which should be resolved somehow, as one can multiply ρ^{arith} by a one-dimensional motive over $\mathbf{k}$ with coefficients in $\overline{\mathbb{Q}}$.

$$X_n := \mathrm{Hom}_{\mathbb{Q}-alg}(\phi_n(A), \overline{\mathbb{Q}})$$

for each $n \geq 1$. This leads to a natural addition to Conjecture 5. Namely, we expect that systems of $\overline{\mathbb{Q}}$-valued functions on X_n associated with higher cohomology spaces arise from elements in $\mathrm{Hom}_{\mathcal{C}_{\mathbb{F}_q}}(\mathbf{1}, A)$ (i.e. from motivic functions on the constructible set underlying algebra A).

More generally, one can expect that the motivic constructible sheaves with some kind of boundedness will be parametrized by commutative algebras.

Formulas from the example from Section 0.1 make sense and give an algebra in $\mathcal{C}_{\mathbf{k}}$ for arbitrary field $\mathbf{k}$. This leads to

Question 12. Can one construct algebras $A_{Y,D,N}$ for arbitrary ground field $\mathbf{k}$, not necessarily finite? In what sense will these algebras "parameterize" motivic local system?

In general, it seems that the natural source of commutative algebras in $\mathcal{C}_{\mathbf{k}}$ is not the representation theory, but (quantum) algebraic integrable systems.

2.4 Towards integrable systems over local fields

Here we will describe briefly an analog of commutative algebras of integral operators as above for arbitrary local fields, i.e. $\mathbb{R}$, $\mathbb{C}$, or finite extensions of $\mathbb{Q}_p$ or $\mathbb{F}_p((x))$. Let us return to our basic example. The check of the associativity of the multiplication law given by formula from Section 0.1 in the case of finite fields is reduced to an identification of certain varieties. The most non-trivial part is the following

Proposition 13. *For generic parameters t, x_1, x_2, x_3, x_4 the two elliptic curves*

$$E: \ f_t(x_1, x_2, y) = w_{12}^2, \ f_t(y, x_3, x_4) = w_{34}^2$$

$$\tilde{E}: \ f_t(x_1, x_3, \tilde{y}) = \tilde{w}_{13}^2, \ f_t(\tilde{y}, x_2, x_4) = \tilde{w}_{24}^2$$

given by equations in variables (y, w_{12}, w_{34}) and $(\tilde{y}, \tilde{w}_{13}, \tilde{w}_{24})$ respectively, are canonically isomorphic over the ground field. Moreover, one can choose such an isomorphism which identifies the abelian differentials

$$\frac{dy}{w_{12}w_{34}} \ \textit{and} \ \frac{d\tilde{y}}{\tilde{w}_{13}, \tilde{w}_{24}} \ .$$

In fact, it is enough to check the proposition over an algebraically closed field and observe that the curves $E, \tilde{E}$ have points over the ground field[11].

Let now $\mathbf{k}$ be a local field. For a given $t \in \mathbf{k} \setminus \{0, 1\}$ we define a (non-negative) half-density c_t on $\mathbf{k}^3$ by the formula

$$c_t := \pi_* \left(\frac{|dx_1|^{1/2} |dx_2|^{1/2} |dx_3|^{1/2}}{|w|} \right)$$

[11] Curve E has 16 rational points with coordinate $y \in \{0, 1, t, \infty\}$, same for $\tilde{E}$.

where

$$\pi : Z(\mathbf{k}) \to \mathbb{A}^3(\mathbf{k}),\ \ \pi(x_1, x_2, x_3, w) = (x_1, x_2, x_3)$$

is the projection of the hypersurface

$$Z \subset \mathbb{A}^4_{\mathbf{k}} :\ f_t(x_1, x_2, x_3) = w^2\ \ .$$

We will interpret c_t as a half-density on $(\mathbb{P}^1(\mathbf{k}))^3$ as well.

One can deduce from the above Proposition the following

Theorem 14. *The operators $T_x,\ x \in \mathbf{k} \setminus \{0, 1, t\}$ on the Hilbert space of half-densities on $\mathbb{P}^1(\mathbf{k})$, given by*

$$T_x(\phi)(y) = \int_{z \in \mathbb{P}^1(\mathbf{k})} c_t(x, y, z)\ \phi(z)$$

are commuting compact self-adjoint operators.

Moreover, in the non-archimedean case one can show that the joint spectrum of commuting operators as above is discrete and consists of densities locally constant on $\mathbb{P}^1(\mathbf{k}) \setminus \{0, 1, t, \infty\}$. In particular, all eigenvalues of operators T_x are algebraic complex numbers. Passing to the limit over finite extensions of $\mathbf{k}$ we obtain a countable set upon which acts

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})\ \ .$$

Also notice that in the case of local fields the formula is much simpler then the motivic one, there is no correction terms. On the other hand, one has a new ingredient, the local density of an integral operator. In general, one can imagine a new formalism[12] where the structure of an algebra is given by data (X, Z, π, ν) where X is a (birational type of) variety over a given field $\mathbf{k}$, Z is another variety, $\pi : Z \to X^3$ is a map (defined only at the generic point of Z), and ν is a rational section of line bundle $K_Z^{\otimes 2} \otimes \pi^*(K_{X^3}^{\otimes -1})$. If $\mathbf{k}$ is a local field then the pushdown by π of $|\nu|^{1/2}$ is a half-density on X^3. The condition of the associativity would follow from a property of certain data formulated purely in terms of birational algebraic geometry.

Presumably, the spectrum for the case of the finite field is just a "low frequency" part of much larger spectrum for p-adic fields, corresponding to some mysterious objects[13].

The commuting integral operators in the archimedean case $\mathbf{k} = \mathbb{R}, \mathbb{C}$ are similar to ones found recently in the usual quantum algebraic integrable systems, see [5].

[12] A somewhat similar formalism was proposed by Braverman and Kazhdan (see [1], who had in mind orbital integrals in the usual local Langlands correspondence.

[13] It looks that all this goes beyond motives, and on the automorphic side is related to some kind of Langlands correspondence for two (or more)-dimensional mixed local-global fields.

3 Third proposal: lattice models

3.1 Traces depending on two indices

Let X be a constructible set over $\mathbb{F}_q$ and M be an endomorphism of X in the category $\mathcal{C}_{\mathbb{F}_q}$ (like e.g. a Hecke operator). What kind of object can be called the "spectrum" of M?

Applying the functors ϕ_n for $n \geq 1$ we obtain an infinite sequence of finite matrices, of exponentially growing size. We would like to understand the behavior of spectra of operators $\phi_n(M)$ as $n \to +\infty$. A similar question arises in some models in quantum physics where one is interested in the spectrum of a system with finitely many states, with the dimension of the Hilbert space depending exponentially on the "number of particles".

Spectrum of an operator acting on a finite-dimensional space can be reconstructed from traces of all positive powers. This leads us to the consideration of the following collection of numbers

$$Z_M(n,m) := \mathrm{Trace}((\phi_n(M))^m)$$

where $n \geq 1$ and $m \geq 0$ are integers. It will be important later to restrict attention only to strictly positive values of m, which mean that we are interested only in non-zero eigenvalues of matrices $\phi_n(M)$, and want to ignore the multiplicity of the zero eigenvalue.

Observation 1. For a given $n \geq 1$ there exists a finite collection of non-zero complex numbers (λ_i) such that for any $m \geq 1$ one has

$$Z_M(n,m) = \sum_i \lambda_i^m \ .$$

Observation 2. For a given $m \geq 1$ there exists a finite collection of non-zero complex numbers (μ_j) and signs $(\epsilon_j \in \{-1,+1\})$, such that for any $n \geq 1$ one has

$$Z_M(n,m) = \sum_j \epsilon_j \mu_j^n \ .$$

The symmetry between parameters n and m (modulo a minor difference with signs) is quite striking.

The first observation is completely trivial. For a given n the numbers (λ_i) are all non-zero eigenvalues of the matrix $\phi_n(M)$.

Let us explain the second observation. By functoriality we have

$$Z_M(n,m) = \mathrm{Trace}(\phi_n(M^m)) \ .$$

Let us assume first that M is given by a constructible set Y which maps to $X \times X$:

$$Y \to X \times X, \ \ y \mapsto (\pi_1(y), \pi_2(y)) \ .$$

Then M^m is given by the consecutive fibered product

$$Y^{(m)} = Y \times_X Y \times_X \cdots \times_X Y \subset Y \times \cdots \times Y$$

of m copies of Y:

$$Y^{(m)}(\overline{\mathbb{F}}_q) = \{(y_1, \ldots, y_m) \in (Y(\overline{\mathbb{F}}_q))^m | \, \pi_2(y_1) = \pi_1(y_2), \ldots, \pi_2(y_{m-1}) = \pi_1(y_m)\}$$

The projection to $X \times X$ is given by $(y_1, \ldots, y_m) \mapsto (\pi_1(y_1), \pi_2(y_m))$. To take the trace we should intersect $Y^{(m)}$ with the diagonal. The conclusion is that $Z_M(n, m)$ is equal to the number of $\mathbb{F}_{q^n}$-points of the constructible set

$$\widetilde{Y}^{(m)} := Y^{(m)} \times_{X \times X} X \ ,$$

$$\widetilde{Y}^{(m)}(\overline{\mathbb{F}}_q) = \{(y_1, \ldots, y_m) \in Y^{(m)}(\overline{\mathbb{F}}_q) | \, \pi_1(y_1) = \pi_2(y_m)\} \ .$$

The second observation is now an immediate corollary of the Weil conjecture on numbers of points of varieties over finite fields[14]. The general case when M is given by a formal *integral* linear combination $\sum_\alpha n_\alpha [Y_\alpha \to X \times X]$ can be treated in a similar way.

3.2 Two-dimensional translation invariant lattice models

There is another source of numbers depending on two indices with a similar behavior with respect to each of indices when another one is fixed. It comes from the so-called lattice models in statistical physics. A typical example is the Ising model. There is a convenient way to encode Boltzmann weights of a general lattice model on $\mathbb{Z}^2$ in terms of linear algebra.

Definition 15. Boltzmann weights of a 2-dimensional translation invariant lattice model are given by a pair V_1, V_2 of finite-dimensional vector spaces over $\mathbb{C}$ and a linear operator

$$R : V_1 \otimes V_2 \to V_1 \otimes V_2 \ .$$

Such data give a function (called the partition function) on a certain set of graphs. Namely, let Γ be a finite oriented graph whose edges are colored by $\{1, 2\}$ in such a way that for every vertex v there are exactly two edges colored by 1 and 2 with head v, and also there are exactly two edges colored by 1 and 2 with tail v. Consider the tensor product of copies of R labelled by the set $Vert(\Gamma)$ of vertices of Γ. It is an element $v_{R,\Gamma}$ of the vector space

$$(V_1^\vee \otimes V_2^\vee \otimes V_1 \otimes V_2)^{\otimes Vert(\Gamma)} \ .$$

The structure of an oriented colored graph gives an identification of the above space with

$$(V_1 \otimes V_1^\vee)^{\otimes Edge_1(\Gamma)} \otimes (V_2 \otimes V_2^\vee)^{\otimes Edge_2(\Gamma)}$$

[14]Here we mean only the fact that the zeta-function of a variety over is rational in q^s, and not the more deep statement about the norms of Weil numbers.

where $Edge_1(\Gamma), Edge_2(\Gamma)$ are the sets of edges of Γ colored by 1 and by 2. The tensor product of copies of the standard pairing gives a linear functional u_Γ on the above space. We define the *partition function* of the lattice model on Γ as

$$Z_R(\Gamma) = u_\Gamma(v_{R,\Gamma}) \in \mathbb{C} \ \ .$$

An oriented colored graph Γ as above is the same as a finite set with two permutations τ_1, τ_2. The set here is $Vert(\Gamma)$, and permutations τ_1, τ_2 correspond to edges colored by 1 and 2 respectively.

In the setting of *translation invariant* 2-dimensional lattice models we are interested in the values of the partition function only on graphs corresponding to pairs of commuting permutations. Such a graph (if it is non-empty and connected) corresponds to a subgroup $\Lambda \subset \mathbb{Z}^2$ of finite index. We will denote the partition function[15] of the graph corresponding to Λ by $Z_R^{lat}(\Lambda)$.

Finally, Boltzmann data make sense in an arbitrary rigid tensor category $\mathcal{C}$. The partition function of a graph takes values in the commutative ring $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$. In particular, one can speak about *super* Boltzmann data for the category $Super_{\mathbb{C}}$ of finite-dimensional complex super vector spaces.

Transfer matrices

Let us consider a special class of lattices $\Lambda \subset \mathbb{Z}^2$ depending on two parameters. Namely, we set

$$\Lambda_{n,m} := \mathbb{Z} \cdot (n, 0) \oplus \mathbb{Z} \cdot (0, m) \subset \mathbb{Z}^2 \ \ .$$

Proposition 16. *For any Boltzmann data (V_1, V_2, R) and a given $n \geq 1$ there exists a finite collection of non-zero complex numbers (λ_i) such that for any $m \geq 1$ one has*

$$Z_R^{lat}(\Lambda_{n,m}) = \sum_i \lambda_i^m \ \ .$$

The proof is the following. Let us introduce a linear operator (called the *transfer matrix*) by formula:

$$T_{(2),n} := \mathrm{Trace}_{V_1^{\otimes n}}((\sigma_n \otimes id_{V_2^{\otimes n}}) \circ R^{\otimes n}) \in \mathrm{End}(V_2^{\otimes n})$$

where $\sigma_n \in \mathrm{End}(V_1^{\otimes n})$ is the cyclic permutation. Here we interpret $R^{\otimes n}$ as an element of

$$(V_1^\vee)^{\otimes n} \otimes (V_2^\vee)^{\otimes n} \otimes V_1^{\otimes n} \otimes V_2^{\otimes n} = \mathrm{End}(V_1^{\otimes n}) \otimes \mathrm{End}(V_2^{\otimes n}) \ \ .$$

It follows from the definition of the partition function that

$$Z_R^{lat}(\Lambda_{n,m}) = \mathrm{Trace}\left(T_{(2),n}\right)^m$$

[15] In physical literature it is called the partition function with periodic boundary conditions.

for all $m \geq 1$. The collection (λ_i) is just the collection of all *non-zero* eigenvalues of $T_{(2),n}$ taken with multiplicities.

Similarly, one can define transfer matrices $T_{(1),m}$ such that $Z_R^{lat}(\Lambda_{n,m}) = \mathrm{Trace}\left(T_{(1),m}\right)^n$ for all $n, m \geq 1$. We see that the function $(n,m) \mapsto Z_R^{lat}(\Lambda_{n,m})$ has the same two properties as the function $(n,m) \mapsto Z_M(n,m)$ from Section 3.1. For super Boltzmann data one obtains sums of exponents with signs.

3.3 Two-dimensional Weil conjecture

Let us return to the case of an endomorphism $M \in \mathrm{End}_{\mathcal{C}_{\mathbb{F}_q}}(X)$. In Section 3.1 we have defined numbers $Z_M(n,m)$ for $n, m \geq 1$. Results of 3.2 indicate that one should interpret pairs (n,m) as parameters for a special class of "rectangular" lattices in $\mathbb{Z}^2$. A general lattice $\Lambda \subset \mathbb{Z}^2$ depends on 3 integer parameters

$$\Lambda = \Lambda_{n,m,k} = \mathbb{Z} \cdot (n,0) \oplus \mathbb{Z} \cdot (k,m), \ n,m \geq 1, \ 0 \leq k < n \ .$$

Here we propose an extension of function Z_M to all lattices in $\mathbb{Z}^2$:

$$Z_M(\Lambda_{n,m,k}) := \mathrm{Trace}((\phi_n(M))^m (\phi_n(\mathrm{Fr}_X))^k)$$

where $\mathrm{Fr}_X \in \mathrm{End}_{\mathcal{C}_{\mathbb{F}_q}}(X)$ is the graph of the Frobenius endomorphism of the scheme X. Notice that $\phi_n(\mathrm{Fr}_X)$ is periodic with period n for any $n \geq 1$.

Proposition 17. *Function Z_M on lattices in $\mathbb{Z}^2$ defined as above, satisfy the following property: for any two vectors $\gamma_1, \gamma_2 \in \mathbb{Z}^2$ such that $\gamma_1 \wedge \gamma_2 \neq 0$ there exists a finite collection of non-zero complex numbers (λ_i) and signs (ϵ_i) such that for any $n \geq 1$ one has*

$$Z_M(\mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot n\gamma_2) = \sum_i \epsilon_i \lambda_i^n \ .$$

In other words, the series in formal variable t

$$\exp\left(- \sum_{n \geq 1} Z_M(\mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot n\gamma_2) \cdot t^n/n \right)$$

is rational.

The proof is omitted here, we'll just indicate that it follows from the consideration of the action of the Frobenius element and of cyclic permutations on the (étale) cohomology of spaces $\widetilde{Y}^{(m)}$ introduced in Section 3.1.

Also, it is easy to see that the same property holds for the partition function $Z_R^{lat}(\Lambda_{m,n,k})$ for arbitrary (super) lattice models.[16] The analogy leads to a two-dimensional analogue of the Weil conjecture (the name will be explained in the next section):

[16] In general, one can show that for any lattice model given by operator R, and for any matrix $A \in GL(2, \mathbb{Z})$ there exists another lattice model with operator R' such that for any lattice $\Lambda \subset \mathbb{Z}^2$ one has $Z_R^{lat}(\Lambda) = Z_{R'}^{lat}(A(\Lambda))$.

Conjecture 18. *For any endomorphism $M \in \mathrm{End}_{\mathcal{C}_{\mathbb{F}_q}}(X)$ there exists super Boltzmann data (V_1, V_2, R) such that for any $\Lambda \subset \mathbb{Z}^2$ of finite index one has*

$$Z_M(\Lambda) = Z_R^{lat}(\Lambda) \ .$$

Up to now there is no hard evidence for this conjecture, there are just a few cases where one can construct a corresponding lattice model in an ad hoc manner. For example, it is possible (and not totally trivial) to do that for the case when $X = \mathbb{A}^1_{\mathbb{F}_q}$ and M is the graph of the map $x \to x^c$ where $c \geq 1$ is an integer.

The above conjecture means that one can see matrices $\phi_n(M)$ as analogs of transfer matrices[17]. In the theory of integrable models people are interested in systems where the Boltzmann weights R depends non-trivially on a parameter ρ (spaces V_1, V_2 do not vary), and the horizontal transfer matrices commute with each other

$$[T_{(2),n}(\rho_1), T_{(2),n}(\rho_2)] = 0$$

because of Yang-Baxter equation. Theory of automorphic forms seems to produce families of commuting endomorphisms in category $\mathcal{C}_{\mathbb{F}_q}$, which is quite analogous to the integrability in lattice models. There are still serious differences. First of all, commuting operators in the automorphic forms case depend on discrete parameters whereas in the integrable model case they depend algebraically on continuous parameters. Secondly, the spectrum of a Hecke operator in its n-th incarnation (like $T_x^{(n)}$ in Section 0.1) has typically n-fold degeneracy, which does not happen in the case of the usual integrable models with period n.

3.4 Higher-dimensional lattice models and a higher-dimensional Weil conjecture

Let $d \geq 0$ be an integer.

Definition 19. Boltzmann data of a d-dimensional translation invariant lattice model are given by a collection $V_1, \dots, V_d$ of finite-dimensional vector spaces over $\mathbb{C}$ and a linear operator

$$R : V_1 \otimes \cdots \otimes V_d \to V_1 \otimes \cdots \otimes V_d \ .$$

Similarly, one can define d-dimensional lattice model in an arbitrary rigid tensor category. The partition function is a function on finite sets endowed with the action of the free group with d generators. In particular, for abelian actions, it gives a function $\Lambda \mapsto Z_R^{lat}(\Lambda) \in \mathbb{C}$ on the set of subgroups of finite

[17]At least if one is interested in the non-zero part of spectra. In general, the size of the transfer matrix depends on n as an exact exponent, while the size of $\phi_n(M)$ is a finite alternating sum of exponents.

index in $\mathbb{Z}^d$. Also, for any lattice $\Lambda_{d-1} \subset \mathbb{Z}^d$ of rank $(d-1)$ and a vector $\gamma \in \mathbb{Z}^d$ such that $\gamma \notin \mathbb{Q} \otimes \Lambda_{d-1}$, the function

$$n \geq 1 \mapsto Z_R^{lat}(\Lambda_{d-1} \oplus \mathbb{Z} \cdot n\gamma)$$

is a finite sum of exponents. Analogously, for any d-dimensional lattice model R and any integer $n \geq 1$ there exists its dimensional reduction, periodic with period n in d-th coordinate, which is a $(d-1)$-dimensional lattice model $R_{(n)}$ satisfying the property

$$Z_{R_{(n)}}(\Lambda_{d-1}) = Z_R(\Lambda_{d-1} \oplus \mathbb{Z} \cdot n\, e_d)\,, \quad \forall \Lambda_{d-1} \subset \mathbb{Z}^{d-1}$$

where $e_d = (0, \ldots, 0, 1) \in \mathbb{Z}^d = \mathbb{Z}^{d-1} \oplus \mathbb{Z}$ is the last standard basis vector.

Conjecture 20. *For any $(d-1)$-dimensional lattice model $(X_1, \ldots, X_{d-1}, M)$, $d \geq 1$, in the category $\mathcal{C}_{\mathbb{F}_q}$, there exists a d-dimensional super lattice model $(V_1, \ldots, V_d, R)$ in $Super_{\mathbb{C}}$ such that for any $n \geq 1$ the numerical $(d-1)$-dimensional model $\phi_n(M)$ gives the same partition function on the set of subgroups of finite index in $\mathbb{Z}^{d-1}$ as the dimensional reduction $R_{(n)}$.*

In the case $d = 1$ this conjecture follows from the usual Weil conjecture. Namely, a 0-dimensional Boltzmann data in $\mathcal{C}_{\mathbb{F}_k}$ is just an element

$$M \in \mathrm{End}_{\mathcal{C}_{\mathbb{F}_q}}(\mathbf{1}) = \mathrm{End}_{\mathcal{C}_{\mathbb{F}_q}}(\otimes_{i \in \emptyset} X_i)$$

of the Grothendieck group of varieties over $\mathbb{F}_k$ (or of K_0 of the category of pure motives over $\mathbb{F}_k$ with rational coefficients). The corresponding numerical lattice models $\phi_n(M)$ are just numbers, counting $\mathbb{F}_{q^n}$-points in M. By the usual Weil conjecture these numbers are traces of powers of an operator in a super vector space, i.e. values of the partition function for 1-dimensional super lattice model.

Similarly, for $d = 2$ one gets the 2-dimensional Weil conjecture from the previous section.

Evidence: *p*-adic Banach lattice models

Let K be a complete non-archimedean field (e.g. a finite extension of $\mathbb{Q}_p$). We define a d-dimensional *contracting* Banach lattice model as follows. The Boltzmann data consists of

- $2d$ countable generated K-Banach spaces $V_1^{in}, \ldots, V_d^{in}, V_1^{out}, \ldots, V_d^{out}$,
- a bounded operator $R^{vertices} : V_1^{in} \widehat{\otimes} \ldots \widehat{\otimes} V_d^{in} \to V_1^{out} \widehat{\otimes} \ldots \widehat{\otimes} V_d^{out}$,
- a collection of compact operators $R_i^{edges} : V_i^{out} \to V_i^{in},\ i = 1, \ldots, d$.

Such data again give a function on oriented graphs with colored edges, in the definition one should insert operator R_i^{edges} for each edge colored by index $i,\ i = 1, \ldots, d$. In the case of *finite-dimensional* spaces $(V_i^{in}, V_i^{out})_{i=1,\ldots,d}$ we obtain the same partition function as for a usual finite-dimensional lattice model. Namely, one can set

$$R := \left(\otimes_{i=1}^{d} R_i^{edges}\right) \circ R^{vertices}, \;\; V_i = V_i^{in}, \, \forall i = 1, \dots, d$$

or, alternatively,

$$\tilde{R} := R^{vertices} \circ \left(\otimes_{i=1}^{d} R_i^{edges}\right), \;\; \tilde{V}_i := V_i^{out}, \, \forall i = 1, \dots, d \;\; .$$

In particular, for any contracting Banach model one get a function $\Lambda \mapsto Z_R^{lat}(\Lambda) \in K$ on the set of sublattices of $\mathbb{Z}^d$. This function satisfies the property that for any lattice $\Lambda_{d-1} \subset \mathbb{Z}^d$ of rank $(d-1)$ and a vector $\gamma \in \mathbb{Z}^d$ such that $\gamma \notin \mathbb{Q} \otimes \Lambda_{d-1}$, one has

$$Z_R^{lat}(\Lambda_{n-1} \oplus \mathbb{Z} \cdot n\gamma) = \sum_{\alpha} \lambda_\alpha^n, \;\; \forall n \geq 1$$

where (λ_α) is a (possibly) countable $Gal(\overline{K}/K)$-invariant collection of non-zero numbers in $\overline{K}$ (eigenvalues of transfer operators) whose norms tend to zero. Similarly, one can define super Banach lattice models.

Here we announce a result supporting Conjecture 7, the proof is a straightforward extension of the Dwork method for the proving of the rationality of zeta-function of a variety over a finite field.

Theorem 21. *The Conjecture 7 holds if one allows contracting Banach super lattice models over a finite extension of* $\mathbb{Q}_p$ *where* p *is the characteristic of the finite field* $\mathbb{F}_q$.

3.5 Tensor category $\mathcal{A}$ and the Master Conjecture

Let us consider the following rigid tensor category $\mathcal{A}$. Objects of $\mathcal{A}$ are finite-dimensional vector spaces over $\mathbb{C}$. The set of morphisms $\mathrm{Hom}_{\mathcal{A}}(V_1, V_2)$ is defined as the group K_0 of the category of finite-dimensional representations of the free (tensor) algebra

$$T(V_1 \otimes V_2^\vee) := \bigoplus_{n \geq 0} (V_1 \otimes V_2^\vee)^{\otimes n} \;\; .$$

A representation of the free algebra by operators in a vector space U is the same as an action of its generators on U, i.e. a linear map

$$V_1 \otimes V_2^\vee \otimes U \to U \;\; .$$

Using duality we interpret it as a map

$$V_1 \otimes U \to V_2 \otimes U \;\; .$$

The composition of morphisms is defined by the following formula on generators:

$$[V_1 \otimes U \to V_2 \otimes U] \circ [V_2 \otimes U' \to V_3 \otimes U']$$

is equal to

$$[V_1 \otimes (U \otimes U') \to V_3 \otimes (U \otimes U')]$$

where the expression in the bracket is the obvious composition of linear maps

$$V_1 \otimes U \otimes U' \to V_2 \otimes U \otimes U' \to V_3 \otimes U \otimes U' \ .$$

The tensor product in $\mathcal{A}$ coincides on objects with the tensor product in $Vect_{\mathbb{C}}$, the same for the duality. The formula for the tensor product on morphisms is an obvious one, we leave details to the reader.

Like in Section 2.1 (Question 3), we can ask the following

Question 22. Are there interesting non-trivial projectors in $\mathcal{A}$?[18]

We denote by $\mathcal{A}^{kar}$ the Karoubi closure of $\mathcal{A}$.

There exists an infinite chain of tensor functors $(\phi_n^{\mathcal{A}})_{n\geq 1}$ from $\mathcal{A}$ to the category of finite-dimensional vector spaces over $\mathbb{C}$ given by

$$\phi_n^{\mathcal{A}}(V) := V^{\otimes n}$$

on objects, and by

$$[f : V_1 \otimes U \to V_2 \otimes U] \stackrel{\phi_n}{\longmapsto} \mathrm{Trace}_{U^{\otimes n}}((\sigma_n \otimes id_{V_2^{\otimes n}} f^{\otimes n}) \in \mathrm{Hom}_{Vect_{\mathbb{C}}}(V_1^{\otimes n}, V_2^{\otimes n})$$

on morphisms, where $\sigma_n : U^{\otimes n} \to U^{\otimes n}$ is the cyclic permutation. The cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts by automorphisms of $\phi_n^{\mathcal{A}}$. Moreover, the generator of the cyclic group acting on $V^{\otimes n} = \phi_n^{\mathcal{A}}(V)$ is the image under $\phi_n^{\mathcal{A}}$ of a certain central element Fr_V in the algebra of endomorphisms $\mathrm{End}_{\mathcal{A}}(V)$. This “Frobenius” element is represented by the linear map $\sigma : V \otimes U \to V \otimes U$ where $U := V$ and $\sigma = \sigma_2$ is the permutation. As in the case of $\mathcal{C}_{\mathbb{F}_q}$, for any V the operator $\phi_n^{\mathcal{A}}(\mathrm{Fr}_V)$ is periodic with period n.

Let us introduce a small modification $\mathcal{A}'$ of the tensor category $\mathcal{A}$. Namely, it will have the same objects (finite-dimensional vector spaces over $\mathbb{C}$), the morphism groups will be the quotients

$$\mathrm{Hom}_{\mathcal{A}'}(V_1, V_2) := K_0(T(V_1 \otimes V_2^{\vee}) - \mathrm{mod})/\mathbb{Z} \cdot [triv]$$

where $triv$ is the trivial one-dimensional representation of $T(V_1 \otimes V_2^{\vee})$ given by zero map

$$V_1 \otimes \mathbf{1} \stackrel{0}{\to} V_2 \otimes \mathbf{1}$$

All the previous considerations extend to the case of $\mathcal{A}'$.

Amazing similarities between categories $\mathcal{C}_{\mathbb{F}_q}$ and $\mathcal{A}'$ suggests the following

[18] A similar question about commuting endomorphisms in $\mathcal{A}$ is almost equivalent to the study of finite-dimensional solutions of the Yang-Baxter equation.

Conjecture 23. *For any prime p there exists a tensor functor*

$$\Phi_p : \mathcal{C}_{\mathbb{F}_p} \to \mathcal{A}'^{kar}$$

and a sequence of isomorphisms of tensor functors from $\mathcal{C}_{\mathbb{F}_p}$ to $Vect_{\mathbb{C}}$ for all $n \geq 1$

$$iso_{n,p} : \phi_n^{\mathcal{A}'} \circ \Phi_p \simeq i_{Vect_{\mathbb{Q}} \to Vect_{\mathbb{C}}} \circ \phi_n$$

where $i_{Vect_{\mathbb{Q}} \to Vect_{\mathbb{C}}}$ is the obvious embedding functor from the category of vector spaces over $\mathbb{Q}$ to the one over $\mathbb{C}$. Moreover, for any $X \in \mathcal{C}_{\mathbb{F}_p}$ the functor Φ_p maps the Frobenius element $\mathrm{Fr}_X \in \mathrm{End}_{\mathcal{C}_{\mathbb{F}_p}}(X)$ to $\mathrm{Fr}_{\Phi_p(V)}$.

This conjecture we call the Master Conjecture because it implies simultaneously *all* higher-dimensional versions of the Weil conjecture at once, as one has the bijection (essentially by definition)

$$\{(d-1)\text{-dimensional super lattice models in } \mathcal{A}'\} \simeq$$

$$\simeq \{d\text{-dimensional super lattice models in } Vect_{\mathbb{C}}\} \ .$$

Remark 24. One can consider a larger category $\mathcal{A}^{super}$ adding to objects of $\mathcal{A}$ super vector spaces as well. The group K_0 in the super case should be defined as the naive K_0 modulo the relation

$$[V_1 \otimes U \to V_2 \otimes U] = -[V_1 \otimes \Pi(U) \to V_2 \otimes \Pi(U)]$$

where Π is the parity changing functor.

It suffices to verify the Master Conjecture only on the full symmetric monoidal subcategory of $\mathcal{C}_{\mathbb{F}_p}$ consisting of powers $\left(\mathbb{A}^n_{\mathbb{F}_p}\right)_{n\geq 0}$ of the affine line. The reason is that any scheme of finite type can be embedded (by a constructible map) in an affine space $\mathbb{A}^n_{\mathbb{F}_p}$, and the characteristic function of the image of such an embedding as an idempotent in $\mathrm{End}_{\mathcal{C}_{\mathbb{F}_p}}(\mathbb{A}^n_{\mathbb{F}_p})$.

Machine modelling finite fields

Let us fix a prime p. The object $A := \mathbb{A}^1_{\mathbb{F}_p}$ of $\mathcal{C}_{\mathbb{F}_p}$ is a commutative algebra (as well as any scheme of finite type, see 2.2.1), with the product given by the diagonal in its cube. The category $\mathrm{Aff}(\mathcal{C}_{\mathbb{F}_p})$ of "affine schemes" in $\mathcal{C}_{\mathbb{F}_p}$ (i.e. the category opposite to the category of commutative associative unital algebras in $\mathcal{C}_{\mathbb{F}_p}$) is closed under finite products. In particular, it makes sense to speak about group-like etc. objects in $\mathrm{Aff}(\mathcal{C}_{\mathbb{F}_p})$. Affine line A is a commutative ring-like object in $\mathrm{Aff}(\mathcal{C}_{\mathbb{F}_p})$, with the operations of addition and multiplication corresponding to the graphs of the usual addition and multiplication on $\mathbb{A}^1_{\mathbb{F}_p}$. In plain terms, this means that besides the commutative algebra structure on A

$$m : A \otimes A \to A$$

we have two coproducts (for the addition and for the multiplication)

$$co - a : A \to A \otimes A,\ co - m : A \to A \otimes A$$

which are homomorphisms of algebras, and satisfy the usual bunch of rules for commutative associative rings, including the distributivity law.

If the Master Conjecture 8 is true then it gives an object $V_p := \Phi_p(A) \in \mathcal{A}'^{kar}$, with one product and two coproducts. One can expect that it is just $\mathbb{C}^p$ as a vector space. For any $n \geq 1$ the $\mathcal{A}'$-product on V_p defines a commutative algebra structure on $V_p^{\otimes n}$. Its spectrum should be a finite set consisting of p^n elements. Two coproducts give operations of addition and multiplication on this set, and we will obtain a *canonical* construction[19] of the finite field $\mathbb{F}_{p^n}$ uniformly for all $n \geq 1$.

Even in the case $p = 2$ the construction of such V_p is a formidable task: one should find 3 finite-dimensional super representations of the free algebra in 8 generators, satisfying 9 identities in various K_0 groups.

3.6 Corollaries of the Master Conjecture

Good sign: Bombieri-Dwork bound

One can deduce easily from the Master Conjecture that for any given p and any system of equations in arbitrary number of variables (x_i) where each of equations is of an elementary form like $x_{i_1} + x_{i_2} = x_{i_3}$, or $x_{i_1} x_{i_2} = x_{i_3}$ or $x_i = 1$, the number of solutions of this system over $\mathbb{F}_{p^n}$ is an alternating sum of exponents in n, with the total number of terms bounded by C^N where $C = C_p$ is a constant depending on p, and N is the number of equations. In fact, it is a well-known Bombieri-Dwork bound (and C is an absolute constant[20]), see [2].

Bad sign: cohomology theories for motives over finite fields

Any machine modelling finite field should be defined over a finitely generated commutative ring. In particular, there should be a machine defined over a number field K_p depending only on the characteristic p. A little thinking shows that the enumeration of the number of solutions of any given system of equations in the elementary form as above, will be expressed as a super trace of an operator in a finite-dimensional super vector space defined over K_p. On the other hand, it looks very plausible that the category of motives over any finite field $\mathbb{F}_q$ does not have any fiber functor defined over a number field, see [9] for a discussion. I think that this is a strong sign indicating that the Master Conjecture is just wrong!

---

[19] Compare with question 2 in Section 1.3, and remarks afterwards.

[20] A straightforward application of [2] gives the upper bound $C \leq 17^4$ which is presumably very far from the optimal one.

4 Categorical afterthoughts

4.1 Decategorifications of 2-categories

Two categories, $\mathcal{C}_{\mathbf{k}}$ and $\mathcal{A}$ introduced in this paper have a common feature which is also shared (almost) by the category of Grothendieck motives. The general framework is the following.

Let $\mathcal{B}$ be a 2-category such that for any two objects $X, Y \in \mathcal{B}$ the category of 1-morphisms $Hom_{\mathcal{B}}(X,Y)$ is a small *additive* category, and the composition of 1-morphisms is a bi-additive functor. In practice we may ask for categories $Hom_{\mathcal{B}}(X,Y)$ to be triangulated categories (enriched in their turn by spectra, or by complexes of vector spaces). Moreover, the composition could be only a weak functor (e.g. A_∞-functor), and the associativity of the composition could hold only up to (fixed) homotopies and higher homotopies. The rough idea is that objects of $\mathcal{B}$ are "spaces" (non-linear in general), whereas objects of the category $Hom_{\mathcal{B}}(X,Y)$ are linear things on the "product" $X \times Y$ interpreted as kernels of some additive functors transforming some kind of sheaves from X to Y, by taking the pullback from X, the tensor product with the kernel on $X \times Y$, and then the direct image with compact supports to Y.

In such a situation one can define a new (1-)category $K^{tr}(\mathcal{B})$ which is in fact a triangulated category. This category will be called the *decategorification* of $\mathcal{B}$.

The first step is to define a new 1-category $K(\mathcal{B})$ enriched by spectra. It has the same objects as $\mathcal{B}$, the morphism spectrum $Hom_{K(\mathcal{B})}(X,Y)$ is defined as the spectrum of K-theory of the triangulated category $Hom_{\mathcal{B}}(X,Y)$[21].

The second step is to make a formal triangulated envelope of this category. This step needs nothing, it can be performed for arbitrary category enriched by spectra. Objects of the new category are finite extensions of formal shifts of the objects of $K(\mathcal{B})$, like e.g. twisted complexes by Bondal and Kapranov.

At the third step one adds formally direct summands for projectors. The resulting category $K^{tr}(\mathcal{B})$ is the same as the full category of compact objects in the category of exact functors from $K(\mathcal{B})^{opp}$ to the triangulated category of spectra (enriched by itself).

Finally, one can define a more elementary pre-additive[22] category $K_0(\mathcal{B})$ by defining $Hom_{K_0(\mathcal{B})}(X,Y)$ to be K_0 group of triangulated category $Hom_{\mathcal{B}}(X,Y)$. Then we add formally to it finite sums and images of projectors. The resulting additive Karoubi-closed category will be denoted by $K_0^{kar}(\mathcal{B})$ and called K_0-decategorification of $\mathcal{B}$. In what follows we list several examples of decategorifications.

[21] It is well-known that in order to define a correct K-theory one needs either an appropriate enrichment on $Hom_{\mathcal{B}}(X,Y)$, or a model structure in the sense of Quillen, see e.g. [10].

[22] Enriched by abelian groups in the plain sense (without higher homotopies).

Non-commutative stable homotopy theory

R. Meyer and R. Nest introduced in [8] a non-commutative analog of the triangulated category of spectra. Objects of their category are not necessarily unital C^*-algebras, the morphism group from A to B is defined as the bivariant Kasparov theory $KK(A,B)$. One of main observations in [8] is that this gives a structure of a triangulated category on C^*-algebras. Obviously this construction has a flavor of the K_0-decategorification.

Elementary algebraic model of bivariant K-theory

One can define a toy algebraic model of the construction by Meyer and Nest. For a given base field $\mathbf{k}$ consider the pre-additive category whose objects are unital associative $\mathbf{k}$-algebras, and the group of morphisms from A to B is defined as K_0 of the exact category consisting of bimodules ($A^{op}\otimes B$-modules) which are projective and finitely generated as B-modules. This is obviously a K_0-decategorification of a 2-category.

Non-commutative pure and mixed motives

Let us consider the quotient of the category of Grothendieck Chow motives $Mot_{\mathbf{k},\mathbb{Q}}$ over given field $\mathbf{k}$ with rational coefficients, by an autoequivalence given by the invertible functor $\mathbb{Q}(1)\otimes\cdot$. The set of morphisms in this category between motives of two smooth projective schemes X, Y is given by

$$\mathrm{Hom}_{Mot_{\mathbf{k},\mathbb{Q}}/\mathbb{Z}^{\mathbb{Q}(1)\otimes\cdot}}(X,Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{Mot_{\mathbf{k},\mathbb{Q}}}(X, \mathbb{Q}(n)\otimes Y) =$$

$$= \left(\mathbb{Q}\otimes_{\mathbb{Z}} \bigoplus_{n\in\mathbb{Z}} \mathrm{Cycles}_n(X\times Y)\right) /(\text{ rational equivalence }) =$$

$$= \mathbb{Q}\otimes_{\mathbb{Z}} \bigoplus_{n\in\mathbb{Z}} CH^n(X\times Y) = \mathbb{Q}\otimes_{\mathbb{Z}} K^0(X\times Y)$$

because the Chern character gives an isomorphism modulo torsion between the sum of all Chow groups and $K^0(X) = K_0(D^b(Coh\,X))$, the K_0 group of the bounded derived category $D(X) := D^b(Coh\,X)$ of coherent sheaves on X. Finally, the category $D(X\times Y)$ can be interpreted as the category of functors $D(Y)\to D(X)$.

Triangulated categories of type $D(X)$ where X is a smooth projective variety over $\mathbf{k}$ belong to a larger class of *smooth proper* triangulated $\mathbf{k}$-linear dg-categories (another name is "saturated categories"), see e.g. [7],[12]. We see that the above quotient category of pure motives is a full subcategory of K_0-decategorification (with $\mathbb{Q}$ coefficients) of the 2-category of smooth proper $\mathbf{k}$-linear dg-categories. This construction was described recently (without mentioning the relation to motives) in [11].

Analogously, if one takes the quotient of the Voevodsky triangulated category of mixed motives by the endofunctor $\mathbb{Q}(1)[2]\otimes\cdot$, the resulting triangulated category seems to be similar to a full subcategory of the full decategorification of the 2-category of smooth proper $\mathbf{k}$-linear dg-categories.

Motivic integral operators

We mentioned already in Section 2.1 that the category $\mathcal{C}_{\mathbf{k}}$ should be considered as a K_0-decategorification of a 2-category of motivic sheaves. A similar 2-category was considered in [6] in the relation to questions in integral geometry and calculus of integral operators with holonomic kernels.

Correspondences for free algebras

The category $\mathcal{A}$ is a K_0-decategorification by definition.

4.2 Trace of an exchange morphism

Let G_1, G_2 be two endofunctors of a triangulated category $\mathcal{C}$, and an exchange morphism (a natural transformation)

$$\alpha : G_1 \circ G_2 \to G_2 \circ G_1$$

is given[23]. Under the appropriate finiteness condition (e.g. when $\mathcal{C}$ is smooth and proper) one can define the *trace* of α, which can be calculated in two ways, as the trace of endomorphism of $\mathrm{Tor}(G_1, id_{\mathcal{C}})$ associated with G_2 and α, and as a similar trace with exchanged G_1 and G_2 (see [4] for a related stuff). Passing to powers and natural exchange morphisms constructed from nm copies of α:

$$\alpha_{(n,m)} : G_1^n \circ G_2^m \to G_2^m \circ G_1^n$$

we obtain a collection of numbers $Z_\alpha(n,m) := \mathrm{Trace}(\alpha_{(n,m)})$ for $n, m \geq 1$. It is easy to see that these numbers come from a 2-dimensional super lattice model.

Let $\mathcal{C} = D(X)$ for smooth projective X, and functors are given by F^* and by $\mathcal{E} \otimes \cdot$ where $F : X \to X$ is a map, and $\mathcal{E}$ is a vector bundle endowed with a morphism $g : F^*\mathcal{E} \to \mathcal{E}$ (as in Section 1.3). In this case $Z_\alpha(n,m)$ is the trace (without the denominator) associated with the map F^n and the bundle $\mathcal{E}^{\otimes m}$. For example, one can construct a 2-dimensional super lattice model with the partition function

$$Z_R^{lat}(\Lambda_{n,m}) = \sum_{x \in \mathbb{C}: F^n(x) = x} x^m$$

[23] We do not assume that α is an isomorphism.

where $F : \mathbb{C} \to \mathbb{C}$ is a polynomial map[24], e.g. $F(x) = x^2 + c$.

The conclusion is that two different proposals concerning motivic local systems in positive characteristic: the first (algebraic dynamics) and the third one (lattice models) are ultimately related. It is enough to find the dynamical realization, and then the lattice model will pop out. As it was mentioned already, most probably these two proposals would fail, but they still can serve as sources of analogies.

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[24] This seems to be a new type of integrability in lattice models, different from the usual Yang-Baxter ansatz.