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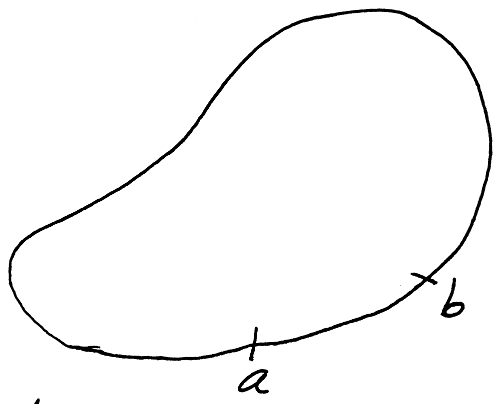
Massless elastic membrane immersed in a 2D viscous incompressible fluid



$\underline{x} = \underline{X}(\theta, t)$

$0 \leq \theta \leq 2\pi$

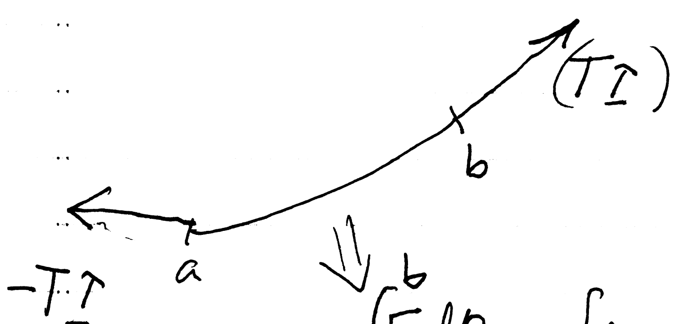
θ = material coordinate

$T(\theta, t)$ = tension in immersed boundary

$$\underline{\tau}(\theta, t) = \frac{\partial \underline{X} / \partial \theta}{|\partial \underline{X} / \partial \theta|} = \text{unit tangent to immersed boundary}$$

$\underline{F}(\theta, t)\, d\theta$ = force applied by arc $d\theta$ of immersed boundary to fluid

Force balance on interval (a, b)



$-\int_a^b \underline{F}\, d\theta$ = force of fluid on boundary

$$0 = T\underline{\tau}\Big|_a^b - \int_a^b \underline{F}\, d\theta$$

$$= \int_a^b \left(\frac{\partial}{\partial\theta}(T\underline{\tau}) - \underline{F} \right) d\theta$$

Since a, b are arbitrary

$$\underline{F} = \frac{\partial}{\partial\theta}(T\underline{\tau})$$

$$= \frac{\partial T}{\partial\theta}\underline{\tau} + T\frac{\partial\underline{\tau}}{\partial\theta}$$

$$= \frac{\partial T}{\partial\theta}\underline{\tau} + TC\underline{n}$$

where

C = curvature

$\underline{n}$ = unit normal to boundary

In general, T is some function of $\left|\frac{\partial \underline{X}}{\partial \theta}\right|$

The special case

$$T = K\left|\frac{\partial \underline{X}}{\partial \theta}\right|$$

is particularly simple. In that case

$$T\underline{\tau} = K\left|\frac{\partial \underline{X}}{\partial \theta}\right| \frac{\partial \underline{X}/\partial \theta}{|\partial \underline{X}/\partial \theta|} = K\frac{\partial \underline{X}}{\partial \theta}$$

so

$$\underline{F} = \frac{\partial}{\partial \theta}(T\underline{\tau}) = K\frac{\partial^2 \underline{X}}{\partial \theta^2}$$

Equations of motion of the whole system in immersed boundary form:

(1) $$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) + \nabla p = \mu\Delta\underline{u} + \underline{f}$$

(2) $$\nabla\cdot\underline{u} = 0$$

(3) $$\underline{f}(\underline{x},t) = \int_0^{2\pi} \underline{F}(\theta,t)\,\delta(\underline{x} - \underline{X}(\theta,t))\,d\theta$$

(4) $$\frac{\partial \underline{X}}{\partial t}(\theta,t) = \underline{U}(\theta,t) = \underline{u}(\underline{X}(\theta,t),t)$$

$$= \int_\Omega \underline{u}(\underline{x},t)\,\delta(\underline{x} - \underline{X}(\theta,t))\,d\underline{x}$$

(5) $$\underline{F}(\theta,t) = K\,\frac{\partial^2 \underline{X}}{\partial\theta^2}(\theta,t)$$

where Ω is the fluid domain.

Equations (1-2) are the Navier-Stokes equations of a viscous incompressible fluid. In these equations

ρ = density, μ = viscosity

$\underline{u}(\underline{x},t)$ = fluid velocity

$p(\underline{x},t)$ = fluid pressure

$\underline{f}(\underline{x},t)$ = force density applied to fluid by immersed boundary

Equations (3-4) are interaction equations. We use the notation

$$\delta(\underline{x}) = \delta(x_1)\,\delta(x_2)$$

where $\underline{x} = (x_1, x_2)$ and $\delta(x)$ is the Dirac delta function

Equation (5) is the immersed boundary equation derived above.

Spatial Discretization

Let: $\{\underline{e}_1, \underline{e}_2\}$ be the standard basis of $\mathbb{R}^2$

$$\mathcal{J}_h = \{\underline{x} : \underline{x} = h(j_1 \underline{e}_1 + j_2 \underline{e}_2), \text{ where } j_1 \text{ and } j_2 \text{ are integers}\}$$

$$(D_\alpha \varphi)(\underline{x}) = \frac{\varphi(\underline{x} + h\underline{e}_\alpha) - \varphi(\underline{x} - h\underline{e}_\alpha)}{2h}$$

$$\underline{D} = (D_1, D_2)$$

$$\underline{D}\varphi = (D_1\varphi, D_2\varphi) \sim \nabla\varphi$$

$$\underline{D}\cdot\underline{u} = D_1 u_1 + D_2 u_2 \sim \nabla\cdot\underline{u}$$

$$(L\underline{u})(\underline{x}) = \sum_{\alpha=1}^{2} \frac{\underline{u}(\underline{x}+h\underline{e}_\alpha) + \underline{u}(\underline{x}-h\underline{e}_\alpha) - 2\underline{u}(\underline{x})}{h^2} \sim \Delta\underline{u}$$

$$S(\underline{u})\varphi = \frac{1}{2}\,\underline{u}\cdot\underline{D}\varphi + \frac{1}{2}\,\underline{D}\cdot(\underline{u}\varphi)$$

$$(S(\underline{u})\,\underline{u})_\alpha = S(\underline{u})u_\alpha$$

Note that $S(\underline{u})\underline{u} \sim \underline{u}\cdot\nabla\underline{u}$ if $\nabla\cdot\underline{u} = 0$

(1') $$\rho\left(\frac{\partial \underline{u}}{\partial t} + S(\underline{u})\underline{u}\right) + \underline{D}p = \mu L \underline{u} + \underline{f}$$

(2') $$\underline{D}\cdot\underline{u} = 0$$

(3') $$\underline{f}(\underline{x},t) = \sum_{k=0}^{N-1} \underline{F}(k\Delta\theta,t)\,\delta_h\left(\underline{x} - \underline{X}(k\Delta\theta,t)\right)\Delta\theta$$

(4') $$\frac{\partial \underline{X}}{\partial t}(k\Delta\theta,t) = \sum_{\underline{x}\in g_h} \underline{u}(\underline{x},t)\,\delta_h\left(\underline{x} - \underline{X}(k\Delta\theta,t)\right)h^2$$

(5') $$\underline{F}(k\Delta\theta,t) = K\,\frac{\underline{X}((k+1)\Delta\theta,t) + \underline{X}((k-1)\Delta\theta,t) - 2\underline{X}(k\Delta\theta,t)}{(\Delta\theta)^2}$$

where

$$\Delta\theta = \frac{2\pi}{N}$$

arithmetic on k is modulo N

Temporal Discretization. $\underline{u}^n(\underline{x}) = \underline{u}(\underline{x}, n\Delta t)$ etc.

Step from $n \to n+1$ begins with preliminary substep from $n \to n+\frac{1}{2}$:

$$\underline{X}^{n+1/2}(k\Delta\theta) = \underline{X}^{n}(k\Delta\theta) + \Delta t \sum_{\underline{x} \in g_h} \underline{u}^n(\underline{x})\, \delta_h\!\left(\underline{x} - \underline{X}^n(k\Delta\theta)\right) h^2$$

$$\underline{F}^{n+1/2}(k\Delta\theta) = K\, \frac{\underline{X}^{n+1/2}((k+1)\Delta\theta) + \underline{X}^{n+1/2}((k-1)\Delta\theta) - 2\underline{X}^{n+1/2}(k\Delta\theta)}{(\Delta\theta)^2}$$

$$\underline{f}^{n+1/2}(\underline{x}) = \sum_{k=0}^{N-1} \underline{F}^{n+1/2}(k\Delta\theta)\, \delta_h\!\left(\underline{x} - \underline{X}^{n+1/2}(k\Delta\theta)\right) \Delta\theta$$

Solve for $\underline{u}^{n+1/2}, \tilde{p}^{n+1/2}$:

$$\left.\begin{aligned} \rho\left(\frac{\underline{u}^{n+1/2} - \underline{u}^n}{\Delta t/2} + S(\underline{u}^n)\underline{u}^n\right) + \underline{D}\tilde{p}^{n+1/2} &= \mu L \underline{u}^{n+1/2} + \underline{f}^{n+1/2} \\ \underline{D}\cdot\underline{u}^{n+1/2} &= 0 \end{aligned}\right\}$$

Complete the step from $n \to n+1$ as follows:

$$\underline{X}^{n+1}(k\Delta\theta) = \underline{X}^{n}(k\Delta\theta) + \Delta t \sum_{\underline{x} \in g_h} \underline{u}^{n+1/2}(\underline{x})\, \delta_h\left(\underline{x} - \underline{X}^{n+1/2}(k\Delta\theta)\right) h^2$$

Solve for $\underline{u}^{n+1}, p^{n+1/2}$:

$$\left.\begin{aligned} &\rho\left(\frac{\underline{u}^{n+1} - \underline{u}^{n}}{\Delta t} + S(\underline{u}^{n+1/2})\underline{u}^{n+1/2}\right) + \underline{D}\, p^{n+1/2} \\ &\qquad = \mu L\left(\frac{\underline{u}^{n} + \underline{u}^{n+1}}{2}\right) + \underline{f}^{n+1/2} \\ &\underline{D} \cdot \underline{u}^{n+1} = 0 \end{aligned}\right\}$$

In both the preliminary substep and the final substep, we need to solve linear systems of the form

$$\left(I - \frac{\Delta t}{2}\frac{\mu}{\rho}L\right)\underline{u} + \frac{\Delta t}{\rho}\underline{D}q = \underline{w}$$

$$\underline{D}\cdot\underline{u} = 0$$

<u>In the preliminary substep:</u>

$$\underline{u} = \underline{u}^{n+1/2}, \quad q = \frac{1}{2}\tilde{p}^{n+1/2},$$

$$\underline{w} = \underline{u}^{n} - \frac{\Delta t}{2} S(\underline{u}^{n})\underline{u}^{n} + \frac{\Delta t}{2\rho}\underline{f}^{n+1/2}$$

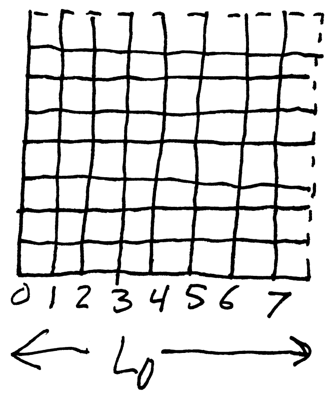
<u>In the final substep:</u>

$$\underline{u} = \underline{u}^{n+1}, \quad q = p^{n+1/2},$$

$$\underline{w} = \underline{u}^{n} - \Delta t\, S(\underline{u}^{n+1/2})\underline{u}^{n+1/2} + \frac{\Delta t}{\rho}\underline{f}^{n+1/2} + \frac{\Delta t}{2}\frac{\mu}{\rho}L\underline{u}^{n}$$

Discrete Fourier Transform solution of the linear system for $(\underline{u}, g)$

Let the fluid domain be $\Omega = (0, L_0) \times (0, L_0)$ with periodic boundary conditions



$h = L/N$

$j_1, j_2 = 0 \cdots N-1$

$$g_h = \{ \underline{x} : \underline{x} = (j_1 h, j_2 h),\ j_\alpha \in \{0, \ldots, N-1\}, \alpha = 1,2 \}$$

Arithmetic on j_1, j_2 is modulo N

For any function $\varphi(\underline{x})$ defined for $\underline{x} \in g_h$,

let $\varphi_{j_1 j_2} = \varphi(j_1 h \underline{e}_1 + j_2 h \underline{e}_2)$

(two-dimensional)

Now define the Discrete Fourier Transform as follows:

$$\hat{\varphi}_{m_1,m_2} = \frac{1}{L_0^2} \sum_{\underline{x} \in g_h} e^{-\frac{2\pi i}{L_0}(m_1 x_1 + m_2 x_2)} \varphi(\underline{x}) h^2$$

$$= \frac{1}{N^2} \sum_{j_1,j_2=0}^{N-1} e^{-\frac{2\pi i}{N}(m_1 j_1 + m_2 j_2)} \varphi_{j_1,j_2}$$

It follows that

$$\varphi_{j_1,j_2} = \sum_{m_1,m_2=0}^{N-1} e^{+\frac{2\pi i}{N}(m_1 j_1 + m_2 j_2)} \hat{\varphi}_{m_1,m_2}$$

which may also be written:

$$\varphi(\underline{x}) = \sum_{m_1,m_2=0}^{N-1} e^{\frac{2\pi i}{L_0}(m_1 x_1 + m_2 x_2)} \hat{\varphi}_{m_1,m_2}$$

Discrete Fourier Transform of D_α, $\alpha = 1,2$

$$(D_1\varphi)(\underline{x}) = \frac{\varphi(\underline{x}+h\underline{e}_1)-\varphi(\underline{x}-h\underline{e}_1)}{2h}$$

$$= \sum_{m_1,m_2=0}^{N-1}\left(\frac{e^{\frac{2\pi i}{L_0}(m_1(x_1+h)+m_2x_2)} - e^{\frac{2\pi i}{L_0}(m_1(x_1-h)+m_2x_2)}}{2h}\right)\hat{\varphi}_{m_1,m_2}$$

$$= \sum_{m_1,m_2=0}^{N-1}\left(\frac{e^{\frac{2\pi i}{L_0}m_1h} - e^{-\frac{2\pi i}{L_0}m_1h}}{2h}\right)e^{\frac{2\pi i}{L_0}(m_1x_1+m_2x_2)}\,\hat{\varphi}_{m_1,m_2}$$

Therefore

$$(\hat{D}_1)_{m_1,m_2} = \frac{2i\sin\left(\frac{2\pi}{L_0}m_1h\right)}{2h} = \frac{i}{h}\sin\left(\frac{2\pi h}{L_0}m_1\right)$$

and similarly

$$(\hat{D}_2)_{m_1,m_2} = \frac{i}{h}\sin\left(\frac{2\pi h}{L_0}m_2\right)$$

Note: As $h \to 0$,

$$(\hat{D}_\alpha)_{m_1,m_2} \longrightarrow \frac{2\pi i}{L_0}m_\alpha = \left(\frac{\partial}{\partial x_\alpha}\right)^{\wedge}$$

Discrete Fourier Transform of L:

$$(L\underline{u})(\underline{x}) = \sum_{\alpha=1}^{2} \frac{\underline{u}(\underline{x}+h\underline{e}_\alpha)+\underline{u}(\underline{x}-h\underline{e}_\alpha)-2\underline{u}(\underline{x})}{h^2}$$

$$= \sum_{m_1,m_2=0}^{N-1} \left(\sum_{\alpha=1}^{2} \frac{e^{\frac{2\pi i}{L_0} h m_\alpha} + e^{-\frac{2\pi i}{L_0} h m_\alpha} - 2}{h^2} \right) \cdot$$

$$\cdot e^{\frac{2\pi i}{L_0}(m_1 x_1 + m_2 x_2)} \quad \hat{\underline{u}}_{m_1,m_2}$$

Therefore

$$\hat{L} = \sum_{\alpha=1}^{2} \frac{2\cos\left(\frac{2\pi h}{L_0} m_\alpha\right) - 2}{h^2}$$

$$= -\frac{2}{h^2} \sum_{\alpha=1}^{2} \left(1 - \cos\left(\frac{2\pi h}{L_0} m_\alpha\right)\right)$$

$$= -\frac{4}{h^2} \sum_{\alpha=1}^{2} \left(\sin\left(\frac{\pi h m_\alpha}{L_0}\right)\right)^2 \longrightarrow -\frac{4\pi^2}{L_0^2} \sum_{\alpha=1}^{2} m_\alpha^2$$

$$= \hat{\Delta}$$

The Discrete Fourier Transform of the equations satisfied by $\underline{u}, q$ is as follows:

$$\left(1 - \frac{\Delta t}{2}\frac{\mu}{\rho}\hat{L}\right)\hat{\underline{u}} + \frac{\Delta t}{\rho}\hat{\underline{D}}\hat{q} = \hat{\underline{W}}$$

$$\hat{\underline{D}} \cdot \hat{\underline{u}} = 0$$

For each m_1, m_2 this is a system of 3 equations in 3 unknowns: $\hat{u}_1, \hat{u}_2, \hat{q}$. The equations for different m_1, m_2 are not coupled to each other!

We can eliminate $\hat{\underline{u}}$ by applying $\hat{\underline{D}}\cdot$ to both sides of the first equation and by making use of $\hat{\underline{D}} \cdot \hat{\underline{u}} = 0$. The result is:

$$\frac{\Delta t}{\rho}\hat{\underline{D}} \cdot \hat{\underline{D}}\,\hat{q} = \hat{\underline{D}} \cdot \hat{\underline{W}}$$

which has the solution

$$\hat{q} = \frac{\hat{\underline{D}} \cdot \hat{\underline{W}}}{\frac{\Delta t}{\rho}\hat{\underline{D}} \cdot \hat{\underline{D}}}$$

Then

$$\hat{\underline{u}} = \left(\hat{\underline{W}} - \frac{\hat{\underline{D}}(\hat{\underline{D}} \cdot \hat{\underline{W}})}{\hat{\underline{D}} \cdot \hat{\underline{D}}}\right) \Big/ \left(1 - \frac{\Delta t}{2}\frac{\mu}{\rho}\hat{L}\right)$$

Writing out the above more explicitly gives

$$\hat{\zeta}_{m_1,m_2} = \frac{\frac{i}{h}\sin\left(\frac{2\pi}{N}\underline{m}\right)\cdot\underline{\hat{W}}_{m_1,m_2}}{-\frac{\Delta t}{\rho h^2}\sin\left(\frac{2\pi}{N}\underline{m}\right)\cdot\sin\left(\frac{2\pi}{N}\underline{m}\right)}$$

$$\underline{\hat{U}}_{m_1,m_2} = \frac{\underline{\hat{W}}_{m_1,m_2} - \dfrac{\sin\left(\frac{2\pi}{N}\underline{m}\right)\sin\left(\frac{2\pi}{N}\underline{m}\right)\cdot\underline{\hat{W}}_{m_1,m_2}}{\sin\left(\frac{2\pi}{N}\underline{m}\right)\cdot\sin\left(\frac{2\pi}{N}\underline{m}\right)}}{1+\frac{\Delta t}{2}\frac{\mu}{\rho}\frac{4}{h^2}\sin\left(\frac{\pi}{N}\underline{m}\right)\cdot\sin\left(\frac{\pi}{N}\underline{m}\right)}$$

where $\underline{m} = (m_1, m_2)$

$$\sin(a\underline{m}) = (\sin(am_1), \sin(am_2))$$

The cases

$$(m_1, m_2) = (0,0),\ (0, N/2),\ (N/2, 0),\ (N/2, N/2)$$

require special consideration. Going back to the original equation, we see that $\hat{g}$ is undefined but plays no role at all, and that $\hat{\underline{u}}$ is given by

$$\hat{\underline{u}}_{m_1,m_2} = \frac{\hat{\underline{W}}_{m_1,m_2}}{1 + \frac{\Delta t}{2}\frac{\mu}{\rho}\frac{4}{h^2}\sin\left(\frac{\pi}{N}\underline{m}\right)\cdot\sin\left(\frac{\pi}{N}\underline{m}\right)}$$

Construction of δ_h:

Let

$$\delta_h(\underline{x}) = \frac{1}{h^2}\,\varphi\left(\frac{x_1}{h}\right)\varphi\left(\frac{x_2}{h}\right)$$

where $\underline{x} = (x_1, x_2)$ and φ has the following properties:

i) φ is continuous

ii) $\varphi(r) = 0$ for $|r| \geq 2$

iii) $$\sum_{i \text{ even}} \varphi(r-i) = \sum_{i \text{ odd}} \varphi(r-i) = \frac{1}{2}, \quad \text{all } r$$

iv) $$\sum_i (r-i)\,\varphi(r-i) = 0, \quad \text{all } r$$

v) $$\sum_i \left(\varphi(r-i)\right)^2 = C, \quad \text{all } r$$

<u>Note</u>: Unlike i, r is a <u>real</u> variable.

"all r" means all <u>real</u> values of r.

How to determine $\varphi(r)$:

Consider $0 \le r \le 1$. Then the nonzero $\varphi(r-i)$ are at most

$$\varphi(r-2),\ \varphi(r-1),\ \varphi(r),\ \varphi(r+1)$$

Therefore, conditions (iii)–(v) reduce to

$$\varphi(r-2) + \varphi(r) = \frac{1}{2}$$

$$\varphi(r-1) + \varphi(r+1) = \frac{1}{2}$$

$$(r-2)\varphi(r-2) + (r-1)\varphi(r-1) + r\varphi(r) + (r+1)\varphi(r+1) = 0$$

$$(\varphi(r-2))^2 + (\varphi(r-1))^2 + (\varphi(r))^2 + (\varphi(r+1))^2 = C$$

To determine C, set $r=0$. Then $\varphi(r-2)=0$, and the above equations reduce to

$$\varphi(0) = \frac{1}{2}$$

$$\left.\begin{array}{r} \varphi(-1) + \varphi(1) = \frac{1}{2} \\ -\varphi(-1) + \varphi(1) = 0 \end{array}\right\} \Rightarrow \varphi(-1) = \varphi(1) = \frac{1}{4}$$

$$C = (\varphi(-1))^2 + (\varphi(0))^2 + (\varphi(1))^2 = \frac{1}{16} + \frac{1}{4} + \frac{1}{16} = \frac{3}{8}$$

With C known, return to the case $0 \le r \le 1$. Make use of the first two equations to simplify the third one:

$$r\left(\varphi(r-2)+\varphi(r-1)+\varphi(r)+\varphi(r+1)\right)$$

$$= 2\varphi(r-2)+\varphi(r-1)-\varphi(r+1)$$

The factor that multiplies r is equal to 1. Thus, we have the system:

$$\varphi(r-2)+\varphi(r) = \frac{1}{2}$$

$$\varphi(r-1)+\varphi(r+1) = \frac{1}{2}$$

$$2\varphi(r-2)+\varphi(r-1)-\varphi(r+1) = r$$

$$(\varphi(r-2))^2+(\varphi(r-1))^2+(\varphi(r))^2+(\varphi(r+1))^2 = \frac{3}{8}$$

Use the first 3 equations to express $\varphi(r-2), \varphi(r-1), \varphi(r+1)$ in terms of $\varphi(r)$:

$$\varphi(r-2) = \frac{1}{2} - \varphi(r)$$

$$\varphi(r-1) = \frac{1}{2}\left[\frac{1}{2}+r-2\varphi(r-2)\right] = \frac{1}{2}\left(r-\frac{1}{2}\right)+\varphi(r)$$

$$\varphi(r+1) = \frac{1}{2}\left[\frac{1}{2}-r+2\varphi(r-2)\right] = \frac{1}{2}\left(-r+\frac{3}{2}\right)-\varphi(r)$$

Substitute the above results into the sum-of-squares equation:

$$\left(\frac{1}{2}-\varphi(r)\right)^2+\left(\frac{1}{2}\left(r-\frac{1}{2}\right)+\varphi(r)\right)^2+\left(\varphi(r)\right)^2$$

$$+\left(\frac{1}{2}\left(-r+\frac{3}{2}\right)-\varphi(r)\right)^2=\frac{3}{8}$$

Collecting like powers of $\varphi(r)$ yields

$$4\left(\varphi(r)\right)^2+(2r-3)\varphi(r)+\frac{1}{2}(r-1)^2=0$$

$$\varphi(r)=\frac{3-2r\pm\sqrt{(3-2r)^2-8(r-1)^2}}{8}$$

$$=\frac{3-2r\pm\sqrt{1+4r-4r^2}}{8}$$

Recall that this holds only for $0\le r\le 1$
Note that

$$\varphi(0)=\frac{3\pm 1}{8},\quad \varphi(1)=\frac{1\pm 1}{8}$$

Since we previously found $\varphi(0)=1/2$ and $\varphi(1)=1/4$, we must choose the + root.

Then, for $0 \le r \le 1$, we have

$$\varphi(r) = \frac{3 - 2r + \sqrt{1 + 4r - 4r^2}}{8}$$

and also

$$\varphi(r-2) = \frac{4}{8} - \varphi(r) = \frac{1 + 2r - \sqrt{1 + 4r - 4r^2}}{8}$$

$$\varphi(r-1) = \frac{4r-2}{8} + \varphi(r) = \frac{1 + 2r + \sqrt{1 + 4r - 4r^2}}{8}$$

$$\varphi(r+1) = \frac{6-4r}{8} - \varphi(r) = \frac{3 - 2r - \sqrt{1 + 4r - 4r^2}}{8}$$

HOMEWORK:

Plot $\varphi(r)$ and $\varphi'(r)$ for $-3 \le r \le 3$

HOMEWORK:

Evaluate the "6-point delta function", which is defined by

$\varphi(r)$ is continuous

$\varphi(r) = 0$ for $|r| \geq 3$

$$\sum_{i \text{ even}} \varphi(r-i) = \sum_{i \text{ odd}} \varphi(r-i) = \frac{1}{2} \quad \text{all } r$$

$$\sum_i (r-i)\,\varphi(r-i) = 0 \quad \text{all } r$$

$$\sum_i (r-i)^2\,\varphi(r-i) = 0 \quad \text{all } r$$

$$\sum_i (r-i)^3\,\varphi(r-i) = 0 \quad \text{all } r$$

$$\sum_i \left(\varphi(r-i)\right)^2 = C \quad \text{all } r$$

Plot $\varphi(r)$ and $\varphi'(r)$ for $-4 \leq r \leq 4$