Homework 2
Course : The Immersed Boundary Method
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1 Model problem

Use the rigid pIB method in 2D to simulate an immersed boundary in the form of a straight line with uniform mass density and with a pivot point on the line that is held at a fixed location in space (see Figure 1).

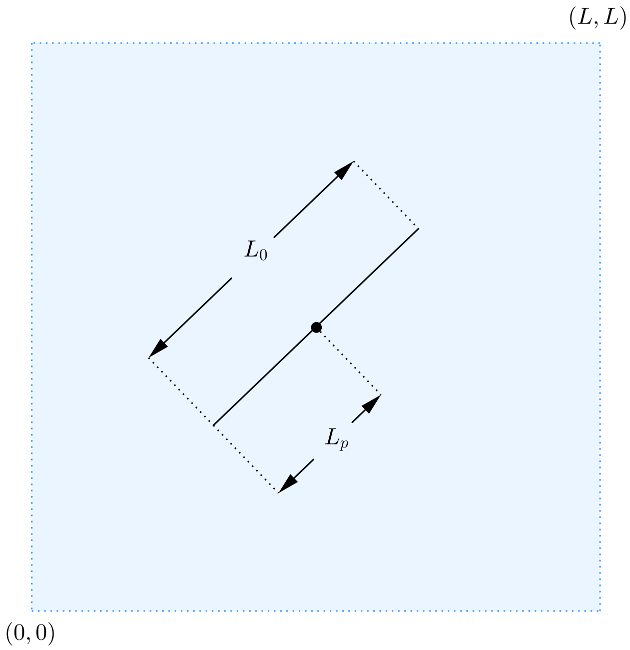

Figure 1: Model problem.

Let the domain be $(0, L) \times (0, L)$ with periodic boundary conditions. Let the flow be driven by an applied force density that is constant in space and time

$$\mathbf{f^0}(\mathbf{x}, t) = (f^0, 0)\,. \tag{1}$$

The pivot point is fixed in space at

$$(L/2, L/2) \tag{2}$$

and the line rotates freely about the pivol point.
Let s denote distance along the line measured from one end. The line has length L_0, so

$$0 \leq s \leq L_0\,. \tag{3}$$

The pivot point is located at $s = L_p$.
The rigid line has only one degree of freedom $\theta(t)$, so its spatial configuration at time t is given by

$$\mathbf{x} = \mathbf{Z}(s, t) = \left(\frac{L}{2} + (s - L_p)\cos(\theta(t)), \frac{L}{2} + (s - L_p)\sin(\theta(t))\right)\,. \tag{4}$$

Because the line boundary has a fixed pivot point, it is better not to make any reference to its center of mass, and instead to define angular momentum, moment of inertia, angular velocity, and torque with

respect to the pivot point. Also, because we are in 2D, we have the simplification that these are all scalar quantities. In particular, the moment of inertia is a number, not a matrix, and it is constant in time even in the laboratory frame. These are major simplifications.
Let m_0 be the mass density of the boundary. Then its moment of inertia is given by

$$\begin{aligned} I_0 &= \int_0^{L_0} m_0 \left(s - L_p\right)^2 \mathrm{d}s \\ &= \frac{m_0}{3}\left(\left(L_0 - L_p\right)^3 + L_p^3\right). \end{aligned} \tag{5}$$

The equations of motion of the whole system, in the rigid pIB formulation, may now be written as follows

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p = \mu\Delta\mathbf{u} + \mathbf{f^0} + \mathbf{f^{IB}} \tag{6}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{7}$$

$$\mathbf{f^{IB}}(\mathbf{x},t) = \int_0^{L_0} K\left(\mathbf{Z}(s,t) - \mathbf{X}(s,t)\right)\delta(\mathbf{x} - \mathbf{X}(s,t))\mathrm{d}s \tag{8}$$

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \int_{(0,L)^2} \mathbf{u}(\mathbf{x},t)\delta(\mathbf{x} - \mathbf{X}(s,t))\mathrm{d}\mathbf{x} \tag{9}$$

$$I_0\frac{d\omega}{dt} = (-\sin\theta, \cos\theta) \bullet \int_0^{L_0} K\left(\mathbf{X}(s,t) - \mathbf{Z}(s,t)\right)(s - L_p)\mathrm{d}s \tag{10}$$

$$\frac{d\theta}{dt} = \omega \tag{11}$$

In these equations $\mathbf{f^0}$ is given by (1), $\mathbf{Z}(s,t)$ is given by (4), and I_0 is given by (5). In particular, at any time t, $\mathbf{Z}(s,t)$ is determined by the value of $\theta(t)$.
The curve $\mathbf{X}(s,t)$ is held close to $\mathbf{Z}(s,t)$ by the spring-like force density

$$K\left(\mathbf{Z}(s,t) - \mathbf{X}(s,t)\right) \tag{12}$$

and at the same time $\mathbf{X}(s,t)$ is moving at the local fluid velocity. This enforces the no-slip condition in the limit $K \to \infty$. The same force density (12) but with opposite sign acts on the rigid line, and the resulting torque about the pivot is evaluated on the right-hand side of (10).

A NOTE ON UNITS

When doing fluid mechanics in 2D, there are two ways to think about the problem:

1. We can think about it as if we are really in a 2D universe. In that case the units are as follows:

$$\begin{aligned} \rho &\sim \text{mass/area} \\ \mu &\sim \text{(mass/area) (lengh}^2\text{/time)} \\ \mathbf{u} &\sim \text{length/time} \\ p &\sim \text{force/length} \\ \mathbf{f} &\sim \text{force/area} \\ m_0 &\sim \text{mass/length} \\ K &\sim \text{(force/length)/length} \end{aligned}$$

2. Alternatively, we can think about the x, y plane of our computation as a cross-section of a 3D problem in which every plane perpendicular to the z-axis is the same and in which there is no velocity in the z direction.

   From this point of view, our rigid "line" is actually the cross section of a rigid planar strip that is infinitely long and has width L_0, and the pivot "point" is actually a pivot axis that is an infinitely long line running parallel to the z-axis.

The equations are exactly the same from either point of view, but the units are different. From the 3D perspective we have more familiar units for fluid quantities, but some quantities like the moment of inertia become the moment of inertia per unit length, where the length is question in the z direction.

From the 3D point of view, we have the following units

$$\begin{aligned}
\rho &\sim \text{mass/volume} \\
\mu &\sim \text{(mass/volume) (lengh}^2\text{/time)} \\
\mathbf{u} &\sim \text{length/time} \\
p &\sim \text{force/area} \\
\mathbf{f} &\sim \text{force/volume} \\
m_0 &\sim \text{mass/area} \\
K &\sim \text{(force/length)/area}
\end{aligned}$$

The task for this homework is to simulate the above system and observe its behaviour for different positions of the pivot and the line (including the center as one special case) and for different boundary mass densities (but note that zero mass density is not possible without a change of methodology, which you might think about).

2 Numerical implementation of the pIB method

We divide the fluid domain $L \times L$ in N^2 squares with side length $h = L/N$, and we divide the rigid boundary in N_b subdivisions with length $\Delta s = L_0/N_b$. The set of the nodes on the fluid mesh is the following

$$\mathbf{g}_h := \{\mathbf{x} : \ \mathbf{x} = (j_1 h \, , \, j_2 h), \ j_\alpha \in \{0, \ldots, N-1\}, \ \alpha = 1, 2\} \, .$$

- We begin with the preliminary substep from $n \to n + 1/2$: For $k = 0, \ldots N_b$,

$$\mathbf{X}^{n+1/2}(k) = \mathbf{X}^n(k) + \frac{\Delta t}{2} \sum_{\mathbf{x} \in \mathbf{g}_h} \mathbf{u}^n(\mathbf{x}) \delta_h(\mathbf{x} - \mathbf{X}^n(k)) h^2 \tag{13}$$

$$\theta^{n+1/2} = \theta^n + \frac{\Delta t}{2} \omega^n \tag{14}$$

$$\mathbf{Z}^{n+1/2}(k) = \left(\frac{L}{2} + (k\Delta s - L_p) \cos(\theta^{n+1/2}), \frac{L}{2} + (k\Delta s - L_p) \sin(\theta^{n+1/2}) \right) \tag{15}$$

$$\begin{aligned}
\tau^{n+1/2} = \; & (-\sin \theta^{n+1/2}, \cos \theta^{n+1/2}) \bullet \\
& \sum_{k=0}^{N_b} \left\{ K \left[(\mathbf{X}^{n+1/2}(k) - \mathbf{Z}^{n+1/2}(k) \right] (k\Delta s - L_p) \right\} \Delta s
\end{aligned} \tag{16}$$

$$\omega^{n+1/2} = \omega^n + \frac{\Delta t}{2} \frac{1}{I_0} \tau^{n+1/2} \tag{17}$$

$$\mathbf{f^{IB}}^{n+1/2}(\mathbf{x}) = \sum_{k=0}^{N_b} K \left(\mathbf{Z}^{n+1/2}(k) - \mathbf{X}^{n+1/2}(k) \right) \delta_h(\mathbf{x} - \mathbf{X}^{n+1/2}(k)) \Delta s \tag{18}$$

Now solve the fluid problem for $\mathbf{u}^{n+1/2}$ and $\tilde{p}^{n+1/2}$, considering the force acting on the fluid ${\mathbf{f}^{\mathbf{IB}}}^{n+1/2} + \mathbf{f}^{\mathbf{0}}$ (see equation (6), and pp 6 of Lecture 1 for the definition of the operators D, S and L):

$$\begin{aligned} \rho\left(\frac{\mathbf{u}^{n+1/2}-\mathbf{u}^n}{(\Delta t)/2} + \mathrm{S}(\mathbf{u}^n)\mathbf{u}^n\right) + \mathrm{D}\tilde{p}^{n+1/2} &= \mu \mathrm{L}\mathbf{u}^{n+1/2} + {\mathbf{f}^{\mathbf{IB}}}^{n+1/2} + \mathbf{f}^{\mathbf{0}} \quad (19) \\ \mathrm{D}\cdot\mathbf{u}^{n+1/2} &= 0 \quad (20) \end{aligned}$$

- The next step is from $n \to n+1$: For $k = 0, \dots N_b$,

$$\begin{aligned} \mathbf{X}^{n+1}(k) &= \mathbf{X}^n(k) + \Delta t \sum_{\mathbf{x}\in\mathbf{g}_h} \mathbf{u}^{n+1/2}(\mathbf{x})\delta_h(\mathbf{x}-\mathbf{X}^{n+1/2}(k))h^2 \quad (21) \\ \theta^{n+1} &= \theta^n + \Delta t\omega^n \quad (22) \\ \omega^{n+1} &= \omega^n + \frac{\Delta t}{I_0}\tau^{n+1/2} \quad (23) \end{aligned}$$

Now solve the fluid problem for $\mathbf{u}^{n+1}$ and $p^{n+1/2}$,

$$\begin{aligned} \rho\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} + \mathrm{S}(\mathbf{u}^{n+1/2})\mathbf{u}^{n+1/2}\right) + \mathrm{D}p^{n+1/2} &= \mu \mathrm{L}\left(\frac{\mathbf{u}^n+\mathbf{u}^{n+1}}{2}\right) + {\mathbf{f}^{\mathbf{IB}}}^{n+1/2} + \mathbf{f}^{\mathbf{0}} \quad (24) \\ \mathrm{D}\cdot\mathbf{u}^{n+1} &= 0 \quad (25) \end{aligned}$$

Remark. *Note that we do not require $\mathbf{Z}^{n+1}$ in the algorithm, but is needed to show the numerical results. We can calculate it in each step time as follows*

$$\mathbf{Z}^{n+1}(k) = \left(\frac{L}{2} + (k\Delta s - L_p)\cos(\theta^{n+1}), \frac{L}{2} + (k\Delta s - L_p)\sin(\theta^{n+1})\right) \quad (26)$$

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