1 American Options

Most traded stock options and futures options are of American-type while most index options are of European-type.

The central issue is when to exercise?

From the holder point of view, the goal is to maximize holder's profit (Note that here the writer has no choice!)

1.1 Some General Relations (for the no dividend case)

The Call Option:

1. $C_A(0) \geq (S(0) - K)_+$

   Proof:

   (1) $C_A(0) \geq 0$ (optionality);

   (2) If $C_A(0) < S(0) - K$ (assuming $S(0) > K$)

   buy the option at $C_A(0)$

   then, exercise immediately. This leads to

   $$\text{profit: } S(0) - K$$

   and

   $$\text{the net profit: } S(0) - K - C_A(0) > 0$$

   which gives rise to an arbitrage opportunity.

   Hence, the no-arbitrage argument yields

   $$C_A(0) \geq (S(0) - K)$$

2. $S(0) \geq C_A(0)$

   Proof:

   If $S(0) < C_A(0)$, buy $S(0)$ and sell $C_A(0)$

   yielding a net profit > 0 at $t = 0$.

   Because the possession of the stock

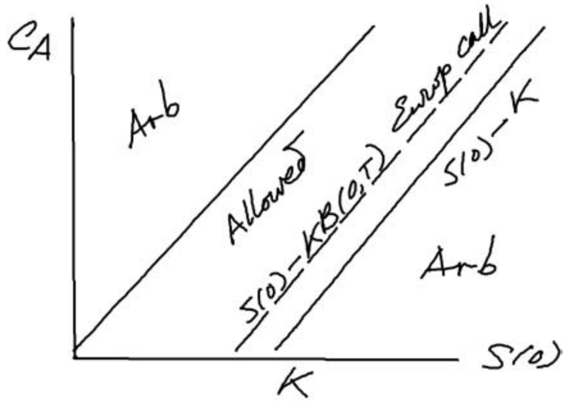
   can always allow the deliverance of the stock

   to cover the exercise if exercised,

   then we are guaranteed to have a **positive** future profit.

   Hence, an arbitrage opportunity.

3. $C_A(0) \geq C_E(0)$ with the same maturity T and strike K.
cf. $C_E(0) \geq (S(0) - KB(0,T))_+$, $\quad \therefore \quad C_A(0) \geq (S(0) - KB(0,T))_+$



4. If the stock has no dividend payment, and the risk-free interest rate is positive, i.e., $B(0,T) < 1 \; \forall T > 0$, then one should **never** prematurely exercise the American call, i.e.,

$$C_A(0) = C_E(0)$$

Proof:

(1) $C_A(0) \geq C_E(0) \geq (S(0) - KB(0,T))_+$ — i.e., the call is "alive"

(2) If exercised now $\Longrightarrow$ the profit $S(0) - K$ — i.e., the call is "dead"

Note that

$$\underbrace{S(0) - KB(0,T)}_{alive} > \underbrace{S(0) - K}_{dead}$$

therefore, it is worth more "alive" than "dead"

Note that

(a) **Question**: Should one exercise the call if $S(0) > K$ and if he believes the stock will go down below K?
No! If exercise,

$$(\text{profit})_1 = S(0) - K$$

If sell the option,

$$(\text{profit})_2 = C_A(0)$$

Since

$$C_A(0) \geq (S(0) - K)_+$$

one should sell the option rather than exercise it!

(b) With dividend, early exercise may be optimal

(c) Intuition — consider paying K to get a stock now vs. paying K to get a stock later, one gets the interest on K, therefore, the difference is

$$Ke^{rT} - K \quad \text{if wait}$$

5. For two American call options, $C_A(t, K, T_1)$ and $C_A(t, K, T_2)$, with the same strike K on the same stock but with different maturities T_1 and T_2, then we have

$$C_A(0, K, T_1) \geq C_A(0, K, T_2)$$

if $T_1 \geq T_2$.

The Put Option:

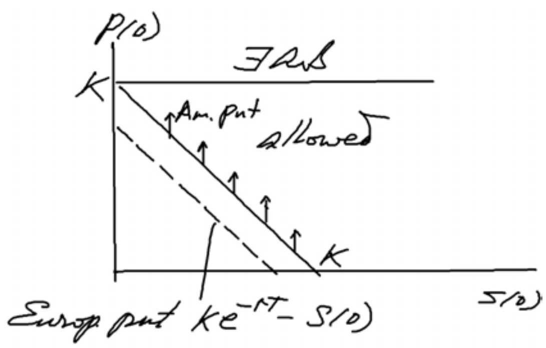
1. $P_A(0) \geq (K - S(0))_+$
   cf. $P_E(0) \geq (KB(0,T) - S(0))_+$
   Proof:

   If $P_A(0) < K - S(0)$,
   buy P_A and exercise immediately, yielding, then, the total cash flow:

$$\underbrace{-P_A(0)}_{\text{buy put}} + \underbrace{(K - S(0))}_{\text{exercise}} > 0$$

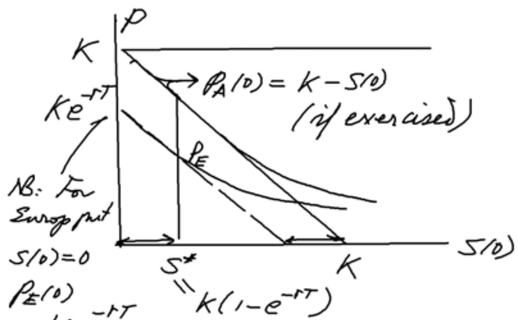
   giving rise to an arbitrage opportunity.

2. $P_A(0) \leq K$



3. $P_A(0) \geq P_E(0)$

   Note that: For a put, the profit is bounded by K. This fact limits the benefit from waiting to exercise and its financial consequence is that one may exercise early if $S(0)$ is very small.



$K \geq P_A \geq (K - S(0))_+$

e.g. For S s.t.

$(K-S)e^{rT} \geq K$

exercise, deposit into bank to grow

exercise at T

$\Longrightarrow$ $(K-S^*)e^{rT} = K$

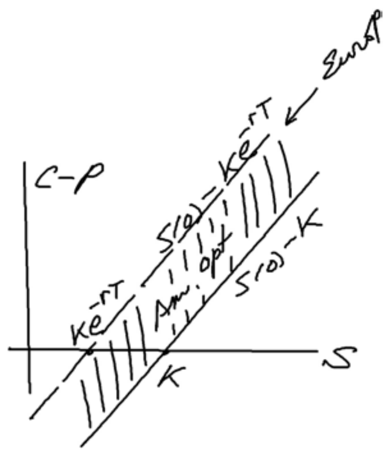
$S^* = K(1-e^{-rT})$

For $S<S^*$, wait to maturity to exercise, only achieve value as the Europ put. which is lower than the total profit if exercised.

at least for $S<S^*$, exercise!

4. Put-call parity for American options:

$$S(0) - K \leq C_A - P_A \leq S_0 - Ke^{-rT}$$



Put-call parity for American options on an non-dividend-paying stock:

(a) $P_A(0) + S(0) - KB(0,T) \geq C_A(0)$;

(b) $C_A(0) \geq P_A(0) + S(0) - K$

i.e.,

$$S(0) - K \leq C_A(0) - P_A(0) \leq S_0 - KB(0,T)$$

Proof:

$$(1) \quad P_A(0) \geq P_E(0) = C_E(0) - S(0) + KB(0,T)$$
$$\because \quad C_E(0) = C_A(0)$$
$$\Longrightarrow \; P_A(0) \geq C_A(0) - S(0) + KB(0,T)$$

(2) Consider portfolio:

long one call
short one put
short the stock
hold K dollars in cash

i.e.,

$$\underbrace{C_A(0)}_{\substack{\text{Never}\\\text{exercised}\\\text{early}}} - \underbrace{P_A(0)}_{\substack{\text{Can be}\\\text{exercised}\\\text{early}}} - S(0) + K$$

If the put is exercised early at t^*, our position is

$$C_A(t^*) - [K - S(t^*)] - S(t^*) + KB(0,t^*)^{-1} = C_A(t^*) + K(B(0,t^*)^{-1} - 1) \geq 0$$

$\Longrightarrow$ liquidated with net positive profit (note that the above inequality holds "$>$" strictly if $S(t^*) > 0$ and $t^* = 0$)

If not exercised earlier, at maturity $t = T$, we have

(i) If $S(T) \leq K$,

$$\text{profit} = 0 - [K - S(T)] - S(T) + KB(0,T)^{-1} = K\left(B(0,T)^{-1} - 1\right) > 0$$

(ii) If $S(T) > K$,

$$\text{profit} = (S(T) - K) - 0 - S(T) + KB(0,T)^{-1} = K\left(B(0,T)^{-1} - 1\right) > 0$$

therefore, the payoff of the portfolio is positive or zero,

$\Longrightarrow$ the present value of the portfolio ≥ 0, i.e.,

$$C_A(0) - P_A(0) - S(0) + K \geq 0$$

Combining (1) and (2) $\Longrightarrow$

$$S(0) - K \leq C_A(0) - P_A(0) \leq S_0 - KB(0,T)$$

$\mathbb{QED}$

Note that: If the stock is dividend-paying, for European options, we have

$$C_E(0) - P_E(0) = P.V.[S(T)] - KB(0,T)$$

where $P.V.[S(T)]$ is the present value of the stock whose price at T is $S(T)$, e.g., If there is a dividend $D(t_1)$ at t_1, then

$$P.V.[S(T)] = S(0) - D(t_1)B(0,t_1)$$

for American options, we have

$$C_A(0) - P_A(0) \leq S_0 - KB(0,T)$$

which is unchanged by dividend, however, in general

$$P.V.[S(T)] - K \leq C_A(0) - P_A(0) \leq S_0 - KB(0,T)$$

1.2 American Calls

1.2.1 Time Value

Consider American calls on no-dividend-paying stocks:

Consider the following strategy: Exercise it at maturity no matter what (obviously, suboptimal if $K > S(T)$), the present value of the American call under this strategy is:

$$P.V.[S(T) - K] = S(0) - KB(0,T)$$

which is equivalent to a forward.

The time value of an American call on a stock without dividends is

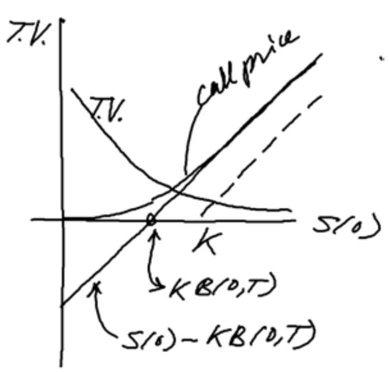
$$T.V.(0) = C_A(0) - [S(0) - KB(0,T)]$$

Note that

$$T.V.(0) \geq 0$$

this is because

$$C_A(0) \geq C_E(0) \geq (S(0) - KB(0,T))_+$$
$$\therefore \quad T.V.(0) \geq 0$$

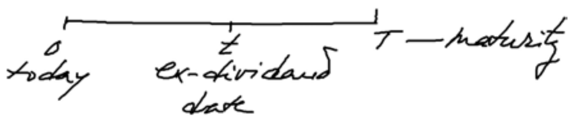


If $S(0) \ll K$, then $T.V.$ is high
If $S(0) \gg K$, then there is a high probability of expiring in-the-money, therefore,

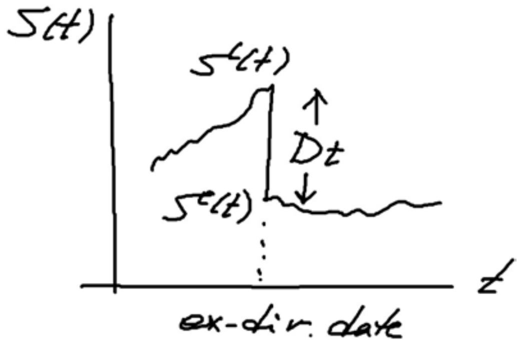
$$C_A(0) \gtrsim S(0) - KB(0,T)$$

$$\text{i.e.,} \quad T.V. \approx 0.$$

1.2.2 Dividends



Result: Given interest rate $r > 0$, it is never optimal to exercise an American call between ex-dividends dates or prior to maturity.



Proof:

Strategy 1: Exercise immediately,

$$(\text{the value})_1 = S(0) - K$$

Strategy 2: Wait till just before the ex-dividend date, and exercise for sure (even if out-of-money)

$$(\text{the value})_2 = \; S^c(t) - K$$

where $S^c(t)$ is the cum stock price just before going ex-dividend. Therefore, the present value is

$$S(0) - KB(0,t)$$

Since $B(0,t) < 1$,

$$\text{the value of Strategy 2} > \text{the value of Strategy 1}$$

therefore, it is best to wait.

Next question: to exercise at anytime after the exdividend date and prior to maturity? The same argument leads to the same conclusion: best to wait.

Question: To exercise or not to exercise?

If exercised just prior to the ex-dividend date,

$$\begin{aligned} \text{the value} &= S(t) - K \\ &= S^e(t) + D_t - K \end{aligned}$$

If not exercised,

$$\begin{aligned} \text{the value} &= C(t) \quad \text{(based on the ex-div stock price)} \\ C(t) &= S^e(t) - KB(t,T) + \underbrace{T.V.(t)}_{\text{Time value at time } t} \end{aligned}$$

Since it should be exercised if and only if the exercised value > the value not exercised, i.e.,

$$S^e(t) + D_t - K > S^e(t) - KB(t,T) + T.V.(t)$$

$\Longrightarrow$

$$D_t > K(1 - B(t,T)) + T.V.(t) > 0 \tag{1}$$

therefore, exercise is optimal at date t iff the dividend is greater than the interest lost on the strike price $K(1 - B(t,T))$ plus the time-value of the call evaluated using the ex-dividend stock price.

Note that

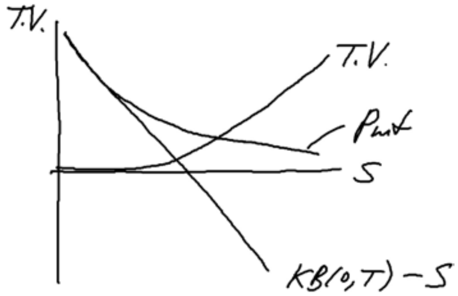
1. If $D_t = 0$ (i.e., no dividend), Eq. (1) does not hold. Hence, never exercise early.
2. Exercise is optimal iff the dividend is large enough (> interest loss + $T.V.$), therefore, if the dividend is small, time-to-maturity is large, it is unlikely to exercise early.

1.3 American Puts

1.3.1 Time Value (if no dividend)

$$T.V.(0) = P_A(0) - \underbrace{[KB(0,T) - S(0)]}_{\substack{\text{the present value} \\ \text{of exercising} \\ \text{the American put} \\ \text{for sure at maturity}}} \geq 0$$

$$P_A(0) \geq P_E(0) \geq (KB(0,T) - S(0))_+$$



If $S(0) \gg K$, then $T.V.$ is large, best to wait
If $S(0) \ll K$, then $T.V.$ is small

1.3.2 Dividend:

Suppose D_t is the dividend per share at time t. The present value of exercising the American put for sure at maturity is

$$P.V. [K - S(T)] = KB(0,T) - [S(0) - D_t B(0,t)]$$

Note that the dividend leads to a stock price drop, hence, added value for the put. The time-value of the put is

$$T.V.(0) = P_A(0) - \{KB(0,T) - [S(0) - D_t B(0,t)]\}$$

To exercise or not to exercise?

1. if exercise: the value is $K - S(0)$

2. if not exercise,

$$P_A(0) = KB(0,T) - [S(0) - D_t B(0,t)] + T.V.(0)$$

   It is optimal to exercise if and only if

$$K - S(0) > P_A(0)$$

   i.e.,

$$K - S(0) > KB(0,T) - [S(0) - D_t B(0,t)] + T.V.(0)$$

   or

$$\underbrace{K(1 - B(0,T))}_{\substack{\text{Interest earned}\\ \text{due to early exercise}}} > \underbrace{D_t B(0,t)}_{\substack{\text{Dividend lost}\\ \text{due to exercise}}} + T.V.(0) \tag{2}$$

   Results:

1. It may be optimal to exercise prematurely even if the stock pays no dividends.

   Proof: If $D_t = 0$, Eq. (2) becomes

$$K(1 - B(0,T)) > T.V.(0)$$

   if $T.V.(0)$ is small, then, early exercise.

2. Dividends tend to delay early exercise.

   Proof: As D_t increases,

$$K(1 - B(0,T)) > D_t B(0,t) + T.V.(0)$$

   may not hold. Hence, to wait.

3. It never pays to exercise just prior to an ex-dividend date.

   Proof: Consider the following two strategies:

   (a) Strategy 1: Exercise just before the ex-dividend date,

$$(\text{value})_1 = K - [S^e(t) + D_t]$$

   Strategy 2: Exercise just after the ex-dividend date,

$$(\text{value})_2 = K - S^e(t)$$

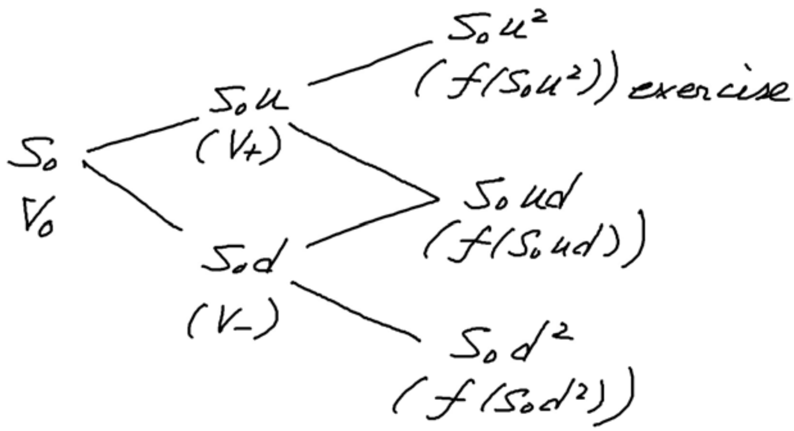
   Since

$$(\text{value})_2 > (\text{value})_1$$

   one should exercise after the ex-dividend date.

1.4 Valuation Using a Binomial Tree

Consider an American option with payoff $f(S_T)$:



$$
\begin{aligned}
u &= e^{\left(r-\frac{\sigma^2}{2}\right)\delta t+\sigma\sqrt{\delta t}} \\
d &= e^{\left(r-\frac{\sigma^2}{2}\right)\delta t-\sigma\sqrt{\delta t}}
\end{aligned}
$$

At the S_0u-node,

$$
\text{The option is worth} \begin{cases} \text{exercise:} & f\left(S_0u\right), & \text{“dead”} \\ \text{not exercise:} & e^{-r\delta t}\left(qf\left(S_0u^2\right)+(1-q)\,f\left(S_0ud\right)\right), & \text{“alive”} \end{cases}
$$

Compare these two values, choose the larger one, i.e., the value is

$$
V_+ = \max\left\{f\left(S_0u\right), e^{-r\delta t}\left(qf\left(S_0u^2\right)+(1-q)\,f\left(S_0ud\right)\right)\right\}
$$

Similarly, at the S_0d-node,

$$
V_- = \max\left\{\underbrace{f\left(S_0d\right)}_{\text{exercised at } t=\delta t}, \underbrace{e^{-r\delta t}\left(qf\left(S_0ud\right)+(1-q)\,f\left(S_0d^2\right)\right)}_{\text{not exercise at } t=\delta t}\right\}
$$

At $t=0$,

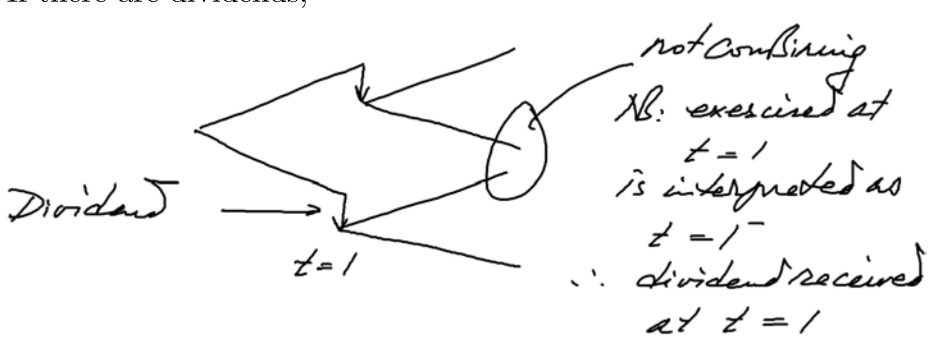
$$
V_0 = \max\left\{\underbrace{f\left(S_0\right)}_{\text{exercised at } t=0}, \underbrace{e^{-r\delta t}\left(qV_+ + (1-q)\,V_-\right)}_{\text{not exercise at } t=0}\right\}
$$

Note that, for an American call,

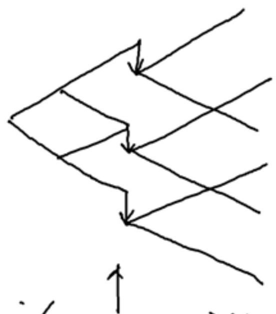
1. If no dividend,

$$
C_A(0) = C_E(0)
$$

2. If there are dividends,



3. Computational complexity:



exponential increase!

if N possible stock prices at ex-dividend date
$\longrightarrow$ N new lattices after the dividend payment.

N.B. **The adaptive mesh methods**: a high resolution (small Δt) tree is grafted onto a low resolution (large Δt) tree. This yields numerical efficiency over regular binomial or trinomial trees. In particular, for American options, there is a need for high resolution close to strike price and to maturity.

1.5 Valuation Using PDE

Consider an American option with an arbitrary payoff $f(S_T)$. $V(S,t)$ denotes its value at time t.

Portfolio:

One American option
Δ shares of the stock

therefore,

$$\Pi = V - \Delta S$$

$$\underbrace{d\Pi - r\Pi dt}_{\text{profit in excess of the risk-free rate}} = \left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - r(V - \Delta S)\right)dt + (V_S - \Delta)\,dS$$

Choose

$$\Delta = \frac{\partial V}{\partial S}$$

then

$$\begin{aligned} d\Pi - r\Pi dt &= \left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - r(V - V_S S)\right)dt \\ &= \left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV\right)dt \\ &\equiv \mathcal{L}_{BS} V\,dt \end{aligned}$$

For a European option:

$$d\Pi - r\Pi dt = 0$$

this is due to no arbitrage, regardless whether it is a long position or a short position.

For an American option, there is an **asymmetry** between a long and a short position since the holder of the right who controls the early-exercise feature.

1. If V is the value of a long position in the American option, the earning is no more than the risk-free rate on our portfolio, i.e.,

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV \leq 0$$

$$\text{if not exercise:} \qquad V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$$

   Why is so? There are 3 possibilities:

$$\mathcal{L}_{BS}V \quad \begin{array}{lll} > 0 & \Longrightarrow & \text{Arbitrage} \\ = 0 & \Longrightarrow & \text{i.e., hold, no arbitrage} \\ < 0 & & \end{array}$$

   Why it is possible to have

$$\mathcal{L}_{BS}V < 0?$$

   This is because we always maximize the holder's profit. If he is not smart enough — not exercise optimally, then, he loses money, the write can make more than risk-free rate. Therefore,

$$\mathcal{L}_{BS}V < 0 \quad \Longrightarrow \quad \text{one should exercise already!}$$

2. If the payoff for early exercise is $f(S,t)$, an no-arbitrage argument leads to

$$V(S,t) \geq f(S,t)$$

   and at maturity,

$$V(S,t) = f(S,t)$$

   which is the final condition.

1.5.1 Specializing to an American put $P(S,t)$:

$$\begin{aligned} \text{Exercise boundary} \quad &: \quad S = S_f(t) \\ S \quad &< \quad S_f(t) \quad \text{— exercise} \\ S \quad &> \quad S_f(t) \quad \text{— not exercise} \end{aligned}$$

At $S = S_f(t)$, there are 3 possibilities:

$$\frac{\partial P}{\partial S} < -1 \qquad (1)$$

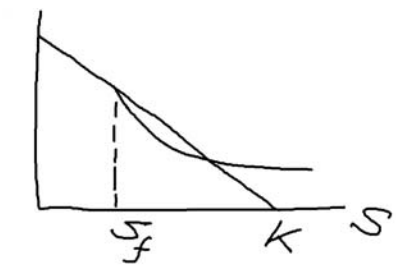
$$\frac{\partial P}{\partial S} > -1 \qquad (2)$$

$$\frac{\partial P}{\partial S} = -1 \qquad (3)$$

Note that, if exercised, the slope of the payoff function is -1 for $S < K$.

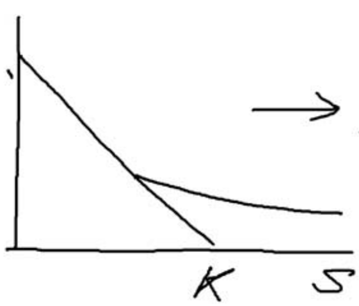
Let's discuss these 3 possibilities:

1. $\dfrac{\partial P}{\partial S} < -1$



violates arbitrage bound
$p(S,t) \geq (S-K)_+$
$\longrightarrow$ (1) is ruled out

2. $\dfrac{\partial P}{\partial S} > -1$



$\longrightarrow$ suboptimal
why?

Why suboptimal?

Consider the strategy adopted by the holder. There are 2 aspects:

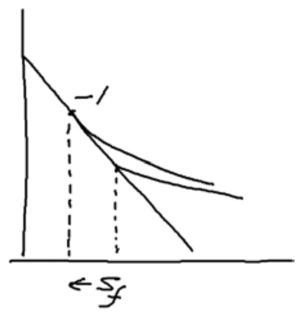
(a) day-to-day arbitrage-based hedging, which leads to the BS equation, i.e., "=" holds,

$$\mathcal{L}_{BS}V = 0$$

(b) Exercise strategy: The boundary condition is

$$P\left(S_f\left(t\right), t\right) = K - S_f\left(f\right)$$

the value of the option near $S = S_f\left(f\right)$ can be increased by choosing a smaller value for S_f, therefore



$$\text{As } P \text{ increases, } \frac{\partial P}{\partial S} \text{ decreases}$$
$$\Longrightarrow \quad \frac{\partial P}{\partial S} = -1 \quad \text{at } S = S_f\left(t\right)$$

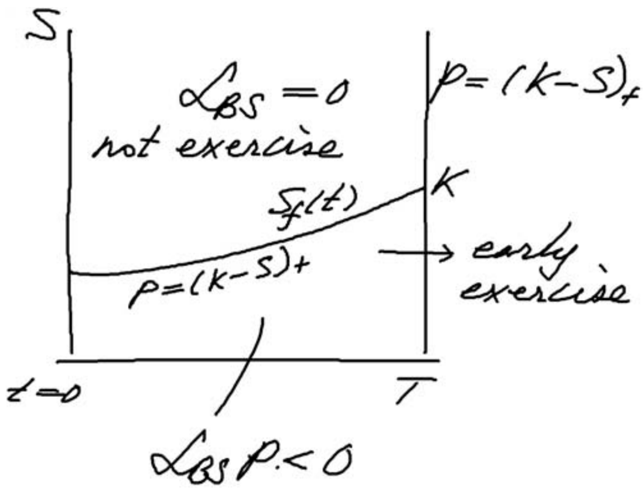
which can be better justified using stochastic control theory, optimal stopping problems or game theory.

Note that the payoff is a solution:

i.e., $\mathcal{L}_{BS}P \leq 0$ when $P = K - S$ for $S < K$. This can be seen by substituting $P = K - S$ into

$$\begin{aligned} & P_t + \frac{1}{2}\sigma^2 S^2 P_{SS} + rsP_S - rP \\ = \; & rS \times (-1) - r\left(K - S\right) \\ = \; & -rK < 0 \\ \therefore \quad & \mathcal{L}_{BS}P \leq 0 \quad \text{for } P = K - S \end{aligned}$$

Now the solution is specified as follows. Dividing S into 2 distinct regions:



1. $0 \le S < S_f(t)$: where an early exercise is optimal:

$$\begin{aligned} P &= K - S \\ \mathcal{L}_{BS}P &< 0 \end{aligned}$$

2. $S_f < S < \infty$: where an early exercise is not optimal:

$$\begin{aligned} P &> K - S \\ \mathcal{L}_{BS}P &= 0 \end{aligned}$$

3. At $S = S_f(t)$,

$$\begin{aligned} P(S_f(f), t) &= (K - S_f(t))_+ \\ \frac{\partial P}{\partial S}(S_f(t), t) &= -1 \end{aligned}$$

   Note that, the second condition is not from the fact that $P(S_f(t), t) = K - S_f(t)$ since there is no a priori known $S_f(t)$.

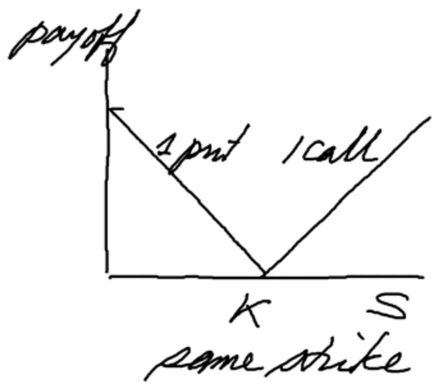
4. Final condition:

$$P = (K - S)_+ \quad \text{at } t = T$$

1.5.2 American options are nonlinear

Note that

1. The pricing of American options is a nonlinear problem due to free boundaries.
   e.g., Consider the perpetual American straddle on a dividend-paying stock:



If this is a *single* contract, then the exercise gives the payoff:

$$(S-K)_+ + (K-S)_+ = |S-K|$$

It is not the same as the sum of the perpetual American put and the perpetual American call. Note that

(a) the "*single*" contract has only one exercise;

(b) The two option contracts have one exercise per option
Hence, the nonlinearity.

2. European-style options are linear.

3. For perpetual American straddles, there are more than one (optimal) free boundary, i.e., when the stock price is too low or too high, one should exercise.

1.6 The Perpetual American Put

1. Payoff:

$$(K-S)_+$$

2. No expiry (we will see that this makes its valuation easier because of time-homogeneity)

Note that:

1. The value of a perpetual American put is time-independent, i.e.,

$$V = V(S)$$

this is due to time-homogeneity — it can be shown that it satisfies time-indepdent BS PDE.

2. Like any American put,

$$V \geq (K - S)_+$$

If not, then $V < (K - S)_+$. We can simply buy the option for price of V and immediately exercise, delivering the stock and receiving K dollars with a net profit

$$K - S - V > 0$$

— Hence, an arbitrage opportunity.

Therefore, the price V is determined by the following mathematical problem:

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 V}{dS^2} + rS\frac{dV}{dS} - rV = 0$$

the general solution of which is

$$V(S) = AS + BS^{-\frac{2r}{\sigma^2}}$$

As $S \to \infty$, the put value $\to 0$. Therefore,

$$A = 0$$

Questioin: How to determine B?

1. If S is high (e.g, $S > K$), not exercise;
2. If S is too low, we should immediately exercise.

Suppose the critical value for this is

$$S = S^*$$

i.e., as soon as S reaches S^* from above, we exercise.

When $S = S^*$,

$$V(S^*) = K - S^*$$

Note that $V(S^*) < K - S^*$ leads to arbitrage while $V(S^*) > K - S^*$ leads to no-exercise.

By continuity of V, we have

$$V(S^*) = B \times (S^*)^{-\frac{2r}{\sigma^2}} = K - S^*$$

therefore,

$$\begin{aligned} B &= (K - S^*)\left(\frac{1}{S^*}\right)^{-\frac{2r}{\sigma^2}} \\ \therefore \quad & V(S) = (K - S^*)\left(\frac{S}{S^*}\right)^{-\frac{2r}{\sigma^2}} \end{aligned}$$

Choose S^* such that V is maximized at any time before exercise, i.e., maximizing our worth (since we can exercise anytime):

$$\frac{\partial}{\partial S^*}\left[(K-S^*)\left(\frac{S}{S^*}\right)^{-\frac{2r}{\sigma^2}}\right]=\frac{1}{S^*}\left(\frac{S}{S^*}\right)^{-\frac{2r}{\sigma^2}}\left(-S^*+\frac{2r}{\sigma^2}(K-S^*)\right)=0$$

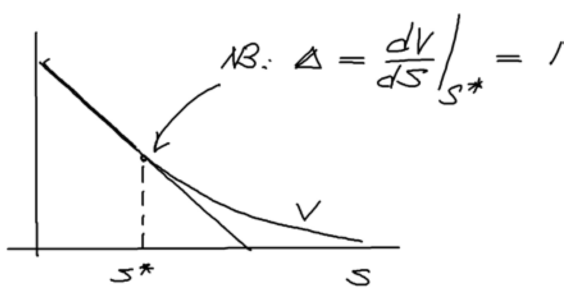
Therefore,

$$S^*=\frac{K}{1+\dfrac{\sigma^2}{2r}}$$

Note that this choice maximizes $V(S)$ for $\forall S>S^*$. Therefore,

$$V(S)=\frac{\sigma^2}{2r}\left(\frac{K}{1+\frac{\sigma^2}{2r}}\right)^{1+\frac{2r}{\sigma^2}}\cdot S^{-\frac{2r}{\sigma^2}}.$$

Note that

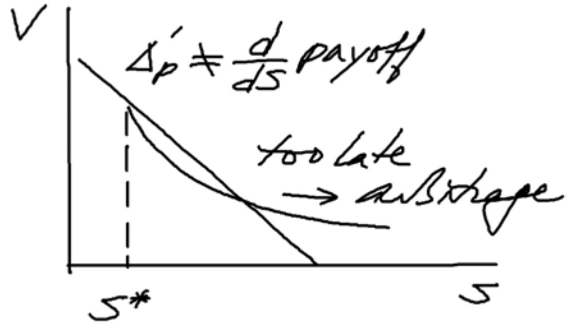


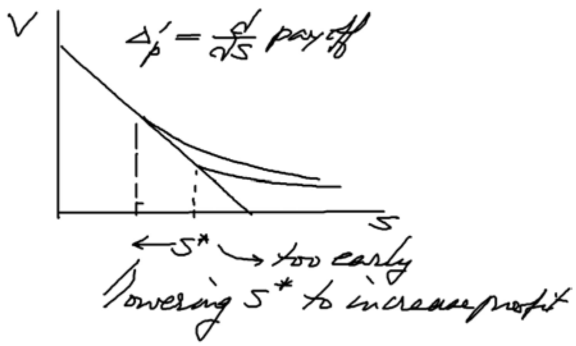
$$\Delta=\left.\frac{dV}{dS}\right|_{S^*}=1=\text{the slope of the payoff}$$

which is the so-called hight-contact or smooth-pasting condition. This is generally true that the American option value is maximized by an exercise strategy that makes the option value and option Delta continuous, leading to exercise the option as soon as the asset price reaches the level at which the option price and payoff meet.

Note that

1. S^* is the optimal exercise point.

2. 

3. 

1.7 Perpetual American Put with Dividend

Assumption: Dividend is continuously paid and has a constant dividend yield on the asset (e.g. the asset is a foreign currency)

Since it is a perpetual option, we have stationarity:

$$\frac{\partial V}{\partial t} = 0$$

therefore,

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + (r - D)\, S\frac{dV}{dS} - rV = 0$$

The general solution has the form:

$$AS^{\alpha^+} + BS^{\alpha^-}$$

where

$$\begin{aligned} \alpha^{\pm} &= \frac{1}{2}\left[-\bar{\mu} \pm \sqrt{\bar{\mu}^2 + \frac{8r}{\sigma^2}}\right] \\ \bar{\mu} &= \frac{2r}{\sigma^2}\left(r - D - \frac{1}{2}\sigma^2\right) \end{aligned}$$

Note that

$$\alpha^- < 0 < \alpha^+$$

A similar analysis gives the followings results: the perpetual American put has value BS^{α^-} with

$$B = -\frac{1}{\alpha^-}\left(\frac{K}{1 - \frac{1}{\alpha^-}}\right)^{1+\alpha^-}$$

and the optimal exercise point is at

$$S^* = \frac{K}{\left(1 - \frac{1}{\alpha^-}\right)}.$$

1.8 Perpetual American Call

The solution has the form

$$
\begin{aligned}
&AS^{\alpha^+} \\
A &= \frac{1}{\alpha^+}\left(\frac{K}{1-\frac{1}{\alpha^+}}\right)^{1+\alpha^+}
\end{aligned}
$$

it is optimal to exercise at

$$S^* = \frac{K}{\left(1-\frac{1}{\alpha^+}\right)}$$

from below. Note that

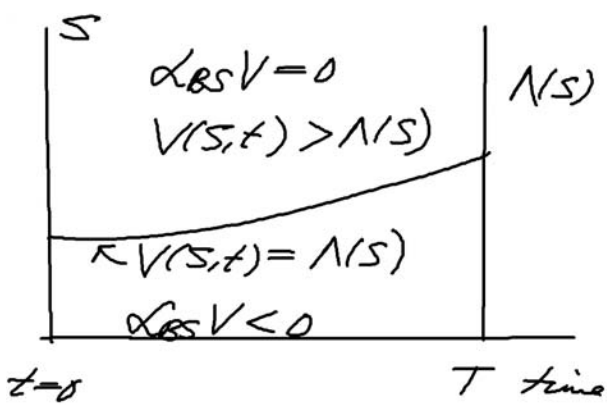
$$
\begin{aligned}
D &= 0 \\
&\Longrightarrow \\
V &= S \quad \text{— just like a stock} \\
\text{and } S^* &= \infty
\end{aligned}
$$

i.e, never optimal to exercise the American perpetual call when there is no dividend (how curious, when do we exercise then?)

1.9 Other American Options

The feature of the option:

1. Payoff $\Lambda(S)$ or $\Lambda(S,t)$.
2. Constant dividend yield.



$$\mathcal{L}_{BS}V = \left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-D)S\frac{\partial}{\partial S} - r\right) V \leq 0$$

when exercise is optimal:

$$\begin{aligned} V(S,t) &= \Lambda(S) \\ \mathcal{L}_{BS}V &< 0 \quad \text{with the strict inequality ``$<$''} \end{aligned}$$

otherwise,

$$\begin{aligned} V(S,t) &> \Lambda(S) \\ \mathcal{L}_{BS}V &= 0 \end{aligned}$$

At the free-boundary,

$$V \text{ and } \frac{\partial V}{\partial S} \text{ are continuous}$$

At maturity,

$$V(S,T) = \Lambda(S).$$

1.9.1 One-touch options:

Recall the European binary option, e.g., binary call:

$$\text{Payoff} = \text{ \$1 if } S_T > K$$

The corresponding American style of this is the so-called one-touch options: which can be exercised anytime for a fixed amount, \$1, if $S_t > K$.

Note that, there is no benefit in holding the option once the level K is reached, leading to immediate exercise as soon as the level is reached first time — hence, the term "one-touch". Reminder: An American option should maximize its value to the holder!

The mathematical feature of the one-touch option is that, since the optimal exercise is determined by the fact that once K is reached for the first time, it reduces a free-boundary problem to a fixed boundary problem.

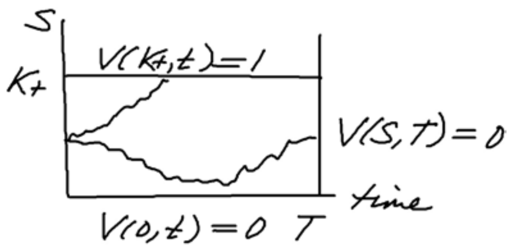
Solving BS PDE with

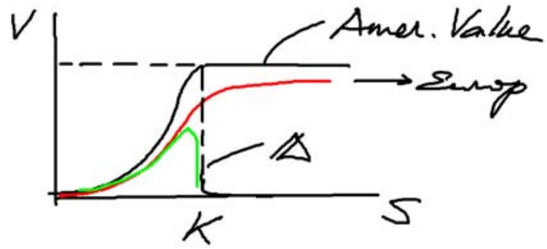
$$\begin{aligned} V(K,t) &= 1 \\ V(S,T) &= 0 \quad \text{for } 0 \le S \le K \end{aligned}$$

yields the solution:

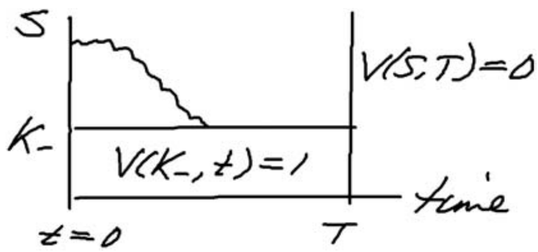
$$\begin{aligned} V(S,t) &= \left(\frac{K_+}{S}\right)^{\frac{2r}{\sigma^2}} N(d_5) + \frac{S}{K_+} N(d_1) \\ d_5 &= \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{S}{K_+}\right) - \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right) \end{aligned}$$

for a one-touch call (see figure).



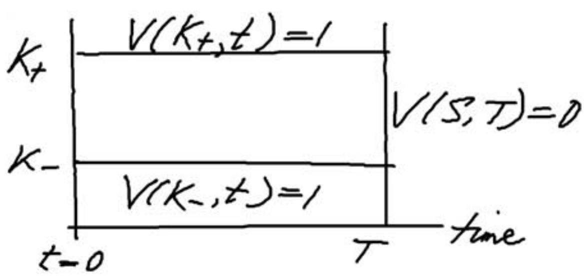


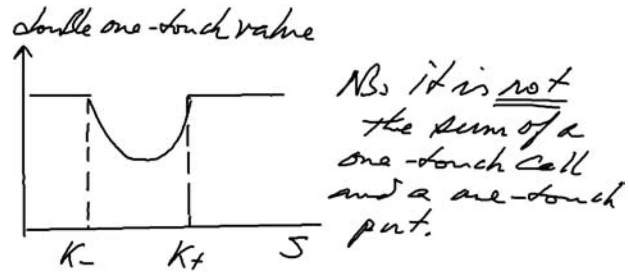
For a one-touch put,



$$
\begin{aligned}
V(S,t) &= \left(\frac{K_-}{S}\right)^{\frac{2r}{\sigma^2}} N(-d_5) + \frac{S}{K_-} N(-d_1) \\
d_5 &= \frac{1}{\sigma\sqrt{T-t}} \left(\log\left(\frac{S}{K_-}\right) - \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right)
\end{aligned}
$$

The double one-touch:





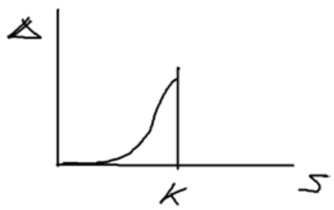
Note that

This American cash-or-nothing (one-touch) is a case with **discontinous** payoffs (cf. the usual requirement that at contact, we have

$$
\begin{aligned}
V(S,t) &= \Lambda(S,t) \quad \text{and} \\
& \Delta \text{ is continous}
\end{aligned}
$$

Since there is no more gain after S crossing K_+ (for a call), no need for hedging for $S > K$. Thus

$$
\Delta = 0 \qquad \text{for } S > K
$$


Δ
K
S