1 Martingales and Their Significance in Option Pricing

1.1 Martingales

$$S_{now} = e^{-r\delta t}\left(qS_{up} + (1-q)\,S_{down}\right)$$

Define

$$X(t) \equiv \frac{S(t)}{e^{rt}}$$

therefore

$$\begin{aligned} X_{now} &= qX_{up} + (1-q)\,X_{down} \\ &= \mathbb{E}_q\left(X_{next}\right) \end{aligned}$$

which is a Martingale.

In general,

$$S(0) = e^{-rT}\mathbb{E}_{RN}\left[S(T)\right]$$

then

$$X(0) = \mathbb{E}_{RN}\left[X(T)\right]$$

Similarly

$$f(0) = e^{-rT}\mathbb{E}_{RN}\left[f(T)\right]$$

Define

$$Y(t) \equiv \frac{f(t)}{e^{rt}}$$

$\Longrightarrow$

$$Y(0) = \mathbb{E}_{RN}\left[Y(T)\right]$$

In general

$$\xi(t) = \mathbb{E}_Q\left[\xi(t)\right], \qquad t' > t$$

defines a Martingale.

Fundamental facts of financial derivatives:

1.

$$\begin{aligned} \text{No Arbitrage} \quad &\Longrightarrow \\ &\exists Q \text{ — some probability measure} \\ \xi \quad &= \quad \frac{\text{value of one option } f}{\text{value of another kind option } g} \\ \text{then,} \quad &\xi \text{ is a Martingale with respect to } Q. \end{aligned}$$

2.

$$\text{No Arbitrage} \iff \exists(\text{a Martingale measure})$$

3.

$$\begin{aligned} &\text{No Arbitrage + Completeness of the market} \\ \iff &\exists!\,(\text{a Martingale measure}) \end{aligned}$$

4. If g is money market account, i.e., $g = e^{rt}$, then Q is the risk-neutral probability.

Basics:
If

$$dy = \alpha(y,t)\,dt + \beta(y,t)\,dW$$

then

$$y \text{ is a martingale} \iff \quad \alpha(y,t) \equiv 0$$

Let us show if $\alpha(y,t) = 0$, then y is a martingale. Since

$$dy = \beta(y,t)\,dW,$$

$$\therefore y(t) - y(0) = \int_0^t \beta(y,s)\,dW(s)$$

$$\mathbb{E}[y(t)] - y(0) = \mathbb{E}\int_0^t \beta(y,s)\,dW(s) = 0 \qquad \text{(N.B. Ito Integral)}$$

$$\therefore \qquad y(0) = \mathbb{E}[y(t)]$$

1.2 Relationship between martingale and risk-neutral processes

We learned before that the risk-neutral process for the stock price movement is

$$dS = rSdt + \sigma SdW.$$

Suppose

$$dS = \alpha dt + \beta dW$$

if

$$\frac{S(t)}{e^{rt}} \text{ is a martingale} \implies \quad \alpha = rS$$

Proof:

$$\begin{aligned} d\left(\frac{S(t)}{e^{rt}}\right) &= d\left(S(t)\,e^{-rt}\right) \\ &= e^{-rt}dS - re^{-rt}Sdt \\ &= e^{-rt}(\alpha dt + \beta dW) - re^{-rt}Sdt \\ &= e^{-rt}(\alpha - rS)\,dt + e^{-rt}\beta dW \end{aligned}$$

Therefore,

$$\alpha = rS$$

i.e., If S is a risk-neutral process, then $e^{-rt}S(t)$ is a martingale.

1.3 Relationship between the BS PDE and Martingales

We have the following fact:

1. If V satisfies the BS PDE, then Ve^{-rt} is a martingale with respect to the risk-neutral measure.
2. No Arbitrage $\Longrightarrow$ Ve^{-rt} is a martingale in the risk-neutral measure.

Proof: (1)

$$\begin{aligned} d\left(Ve^{-rt}\right) &= e^{-rt}dV - re^{-rt}V dt \\ &= e^{-rt}\left(V_t dt + V_S dS + \frac{1}{2}\sigma^2 S^2 V_{SS} dt\right) - re^{-rt}V dt \end{aligned}$$

Substituting the risk-neutral process:

$$dS = rSdt + \sigma S dW$$

leads to

$$\begin{aligned} d\left(Ve^{-rt}\right) &= e^{-rt}\underbrace{\left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV\right)}_{=\text{the BS } \mathcal{L}_{BS}V,\ \therefore \text{it vanishes}} dt + e^{-rt}\sigma S V_S dW \\ &= e^{-rt}\sigma S V_S dW \end{aligned}$$

or

$$d\left(Ve^{-rt}\right) = \sigma S \left(Ve^{-rt}\right)_S dW$$

i.e., the discounted V is a martingale.

(2)

$$\begin{aligned} \text{No Arbitrage} &\Longrightarrow V \text{ is a solution of BS PDE} \\ &\Longrightarrow \frac{V}{e^{rt}} \text{ is a martingale in the risk-neutral measure} \\ &\therefore\ \frac{V(S(0),0)}{e^{r\cdot 0}} = \mathbb{E}_{RN}\left[\frac{V(S(T),T)}{e^{rT}}\right] \\ &V(S(0),0) = e^{-rT}\mathbb{E}_{RN}\left[V(S(T),T)\right] \end{aligned}$$

2 The Market Price of Risk

Consider a natural process:

$$\frac{d\theta}{\theta} = mdt + \Sigma dW$$

where the drift term m is the expected growth rate and Σ is the volatility. $W(t)$ is a Wiener process.

Note that θ need not be the price of an investment asset. For example, it can be the noise level of Times Square or temperature in Beijing.

Suppose we write two derivatives V_1 and V_2 depending on only θ and t, i.e., some options or contracts that give a payoff as a function of θ at some future time.

For simplicity, we assume there is no "dividend", i.e., no income before maturity.

Suppose

$$\frac{dV_1}{V_1} = \mu_1 dt + \sigma_1 dW \quad (1)$$

$$\frac{dV_2}{V_2} = \mu_2 dt + \sigma_2 dW \quad (2)$$

where μ_1, μ_2, σ_1 and σ_2 are functions of time t.

To eliminate risk, we construct the following portfolio:

$$\begin{aligned} \Pi &= \underbrace{(\sigma_2 V_2)}_{\sigma_2 V_2 \text{ units of } V_1} \times \quad V_1 - (\sigma_1 V_1) \times V_2 \\ &= (\sigma_2 - \sigma_1) V_1 V_2 \end{aligned}$$

$$d\Pi = (\sigma_2 V_2)\, dV_1 - (\sigma_1 V_1)\, dV_2$$

Substituting Eqs (1) and (2) into the above equation leads to

$$\begin{aligned} d\Pi &= (\sigma_2 V_2) V_1 (\mu_1 dt + \sigma_1 dW) - (\sigma_1 V_1) V_2 (\mu_2 dt + \sigma_2 dW) \\ &= (\sigma_2 \mu_1 - \sigma_1 \mu_2) V_1 V_2 dt \end{aligned}$$

Since this portfolio is riskless, it must grows at risk-free rate, i.e.,

$$d\Pi = r\Pi dt$$

or

$$(\sigma_2 \mu_1 - \sigma_1 \mu_2) V_1 V_2 = r (\sigma_2 - \sigma_1) V_1 V_2$$

which can be rewritten as

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \equiv \lambda$$

where λ is referred to as the market price of risk of θ. Note that

1. λ is a function of θ and t;

2. λ is independent of V_1 and V_2.

3. The ratio $\frac{\mu-r}{\sigma} = \lambda$ is the same for all derivatives and it depends on θ and t only. We can write

$$\mu - r = \lambda\sigma$$

Note that the LHS is the excess return above the risk-free rate, and σ is a measure of risk (or uncertainty), therefore, λ can be viewed the price of risk, i.e., the excess earning above the risk-free rate per unit σ.

Now we can derive a PDE for any contingent claim V on θ if

$$\frac{dV}{V} = \mu dt + \sigma dW \tag{3}$$

Using Ito's Lemma, we have

$$dV = \left(\frac{\partial V}{\partial t} + \theta m\frac{\partial V}{\partial \theta} + \frac{1}{2}\Sigma\theta^2\frac{\partial^2 V}{\partial \theta^2}\right)dt + \theta\Sigma\frac{\partial V}{\partial \theta}dW \tag{4}$$

Comparing Eq. (3) and Eq. (4) yields

$$\begin{aligned} \mu V &= \frac{\partial V}{\partial t} + \theta m\frac{\partial V}{\partial \theta} + \frac{1}{2}\Sigma\theta^2\frac{\partial^2 V}{\partial \theta^2} \\ \sigma V &= \theta\Sigma\frac{\partial V}{\partial \theta} \end{aligned}$$

or

$$\begin{aligned} \mu &= \frac{1}{V}\left[\frac{\partial V}{\partial t} + \theta m\frac{\partial V}{\partial \theta} + \frac{1}{2}\Sigma\theta^2\frac{\partial^2 V}{\partial \theta^2}\right] \\ \sigma &= \frac{1}{V}\left(\theta\Sigma\frac{\partial V}{\partial \theta}\right) \end{aligned}$$

Since

$$\mu - r = \lambda\sigma$$

$$\therefore \frac{1}{V}\left[\frac{\partial V}{\partial t} + \theta m\frac{\partial V}{\partial \theta} + \frac{1}{2}\Sigma\theta^2\frac{\partial^2 V}{\partial \theta^2}\right] - r = \lambda\frac{1}{V}\left(\theta\Sigma\frac{\partial V}{\partial \theta}\right)$$

i.e.,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\Sigma\theta^2\frac{\partial^2 V}{\partial \theta^2} + \theta\left(m - \lambda\Sigma\right)\frac{\partial V}{\partial \theta} - rV = 0 \tag{5}$$

Note that

1. If θ is some stock price, i.e., a price of of some investment asset, then θ can be viewed as a derivative too. Hence, it must satisfy the relation of market price of risk too:

$$\frac{m-r}{\Sigma}=\lambda$$

i.e.,

$$m-\lambda\Sigma=r$$

Substituting this into Eq. (5), we obtain

$$\frac{\partial V}{\partial t}+\frac{1}{2}\Sigma\theta^2\frac{\partial^2 V}{\partial\theta^2}+\theta r\frac{\partial V}{\partial\theta}-rV=0$$

which is the Black-Scholes equation!

2. In a risk-neutral world,

$$\lambda=0$$

thus, e.g.,

$$\begin{aligned}\frac{dS}{S} &= \mu dt+\sigma dW \qquad \text{and} \quad \frac{\mu-r}{\sigma}=0\\ \therefore &\quad \frac{dS}{S}=rdt+\sigma dW\end{aligned}$$

Example: Bond options.

Stochastic spot rate (spot rate ≡short rate):

$$dr=m(r,t)\,dt+\Sigma(r,t)\,dW \tag{6}$$

e.g.,

1. Ho & Lee model

$$dr=\theta(t)\,dt+\sigma dW$$

2. Vasick/Hull-White model

$$dr=(\theta(t)-\alpha(t)\,r)\,dt+\sigma(t)\,dW$$

3. Cox-Ingersoll-Ross (CIR) model

$$dr=(\theta(t)-\alpha(t)\,r)\,dt+\sigma(t)\sqrt{r}dW$$

4. Black-Karasinki model

$$dr=r\left(\theta(t)-\frac{1}{2}\sigma^2(t)-\alpha(t)\log r\right)dt+r\sigma(t)\,dW$$

Question: How to price a bond?

The issue is how to hedge — unlike a stock, one cannot go out and buy an interest rate, e.g., 10%.

The hedging strategy can be carried out as follows: Use two bonds with different maturities T_1 and T_2 :

$$\begin{array}{lcll} \text{bond } V_1 & : & \text{maturity } T_1 & \quad 1 \text{ unit} \\ \text{bond } V_2 & : & \text{maturity } T_2 & -\Delta \text{ units} \end{array}$$

and the portfolio is

$$\Pi = (V_1 - \Delta V_2)$$

$$\begin{aligned} d\Pi &= \left(\frac{\partial V_1}{\partial t} dt + \frac{\partial V_1}{\partial r} dr + \frac{1}{2} \Sigma^2 \frac{\partial^2 V_1}{\partial r^2} dt \right) \\ &\quad -\Delta \left(\frac{\partial V_2}{\partial t} dt + \frac{\partial V_2}{\partial r} dr + \frac{1}{2} \Sigma^2 \frac{\partial^2 V_2}{\partial r^2} dt \right) \end{aligned}$$

By choosing

$$\Delta = \frac{\dfrac{\partial V_1}{\partial r}}{\dfrac{\partial V_2}{\partial r}}$$

to eliminate the random component, the no arbitrage argument leads to

$$\begin{aligned} d\Pi &= r\Pi dt \\ \text{i.e., } \left[\frac{\partial V_1}{\partial t} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V_1}{\partial r^2} - \frac{\dfrac{\partial V_1}{\partial r}}{\dfrac{\partial V_2}{\partial r}} \left(\frac{\partial V_2}{\partial t} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V_2}{\partial r^2} \right) \right] &= r \left(V_1 - \frac{\dfrac{\partial V_1}{\partial r}}{\dfrac{\partial V_2}{\partial r}} V_2 \right) \end{aligned}$$

$\Longrightarrow$

$$\frac{\dfrac{\partial V_1}{\partial t} + \frac{1}{2}\Sigma^2 \dfrac{\partial^2 V_1}{\partial r^2} - rV_1}{\dfrac{\partial V_1}{\partial r}} = \frac{\dfrac{\partial V_2}{\partial t} + \frac{1}{2}\Sigma^2 \dfrac{\partial^2 V_2}{\partial r^2} - rV_2}{\dfrac{\partial V_2}{\partial r}}$$

Note that the LHS is a function of T_1 not T_2 while the RHS is a function of T_2 not T_1. Therefore, neither side depends on the maturity T, i.e.,

$$\frac{\dfrac{\partial V}{\partial t} + \frac{1}{2}\Sigma^2 \dfrac{\partial^2 V}{\partial r^2} - rV}{\dfrac{\partial V}{\partial r}} = a(r,t)$$

we can write

$$a(r,t) \equiv \Sigma(r,t)\lambda(r,t) - m(r,t)$$

where $\Sigma(r,t)$ and $m(r,t)$ are functions in Eq. (6). For this procedure to hold, we require

$$\Sigma(r,t) \not\equiv 0.$$

Therefore, the PDE for pricing a zero-coupon bond is

$$\begin{aligned}\frac{\partial V}{\partial t}+\frac{1}{2}\Sigma^2\frac{\partial^2 V}{\partial r^2}+(m-\lambda\Sigma)\frac{\partial V}{\partial r}-rV &= 0 \\ \text{with final condition} \quad &: \quad V(r,T)=\$1\end{aligned}$$

Note that

1. $\lambda=\lambda(r,t)$ is a function yet to be determined.
2. If we assume V has the form:

   $$V(r,t)=A(t,T)\,e^{-rB(t,T)}$$

   then

   $$\begin{aligned}\Sigma(r,t) &= (\alpha(t)\,r-\beta(t))^{1/2} \\ m(r,t) &= \left(-\gamma(t)\,r+\eta(t)+\lambda(r,t)\left[\alpha(t)\,r-\beta(t)\right]^{1/2}\right)\end{aligned}$$

   This leads to the following result:

   $$m-\lambda\Sigma=-\gamma(t)\,r+\eta(t)$$

   which is independent of λ!
3. Given those short-rate models above, we have, e.g.,

Vasick	:		$\alpha=0$
CIR	:		$\beta=0$
Hull-White	:	either	$\alpha=0$ or $\beta=0$

   why? cf. Willmott's book.

3 Equivalent Martingale Measures

Recall: If f and g are some derivatives on the same, single process dW,

$$\text{No Arbitrage} \Longrightarrow \frac{f}{g} \text{ is a martingale with respect to some measure}$$

More specific,

$$\begin{aligned} df &= \mu_f f dt + \sigma_f f dW \\ dg &= \mu_g g dt + \sigma_g g dW \end{aligned}$$

which are not necessarily geometric Brownian motions since μ_f and σ_f can depend on f, etc. An no arbitrage argument yields

$$\frac{\mu_f - r}{\sigma_f} = \lambda = \frac{\mu_g - r}{\sigma_g}$$

For a new market price of risk λ^*,

$$\lambda^* = \frac{\mu^* - r}{\sigma}$$

we have

$$df = (r + \lambda^* \sigma_f) f dt + \sigma_f f dW \tag{7}$$

$$dg = (r + \lambda^* \sigma_g) g dt + \sigma_g g dW \tag{8}$$

Note that

1. Market price of risk determines the drift;
2. Volatility does not change;
3. Choosing a drift is equivalent to choosing a market price of risk λ.

From the above argument, we conclude that:

$$\text{No Arbitrage} \implies \frac{f}{g} \text{ is a martingale for some } \lambda$$

The question is which λ. It turns out that

$$\begin{aligned} \text{if } \lambda &= \sigma_g, \\ & \text{then, } \frac{f}{g} \text{ is a martingale for all derivative } f. \end{aligned}$$

This can be demonstrated as follows:

Substituting $\lambda = \sigma_g$ into Eqs. (7) and (8) yields

$$\begin{aligned} df &= (r + \sigma_g \sigma_f) f dt + \sigma_f f dW \\ dg &= \left(r + \sigma_g^2\right) g dt + \sigma_g g dW \end{aligned}$$

Applying Ito's lemma to $\ln f$ and $\ln g$:

$$\begin{aligned} d \ln f &= \left(r + \sigma_g \sigma_f - \frac{1}{2}\sigma_f^2\right) dt + \sigma_f dW \\ d \ln g &= \left(r + \frac{1}{2}\sigma_g^2\right) dt + \sigma_g dW \end{aligned}$$

therefore,

$$\begin{aligned} d\ln\frac{f}{g} &= d\left(\ln f - \ln g\right) \\ &= d\ln f - d\ln g \\ &= -\frac{1}{2}\left(\sigma_f - \sigma_g\right)^2 dt + \left(\sigma_f - \sigma_g\right) dW \end{aligned}$$

Now, we want to use this to compute $d\left(\frac{f}{g}\right)$ by the following method:

If we know

$$\begin{aligned} dX &= \mu_X dt + \sigma_X dW \\ d\ln X &= \mu dt + \sigma dW \end{aligned}$$

what is the relation between (μ_X, σ_X) and (μ, σ)? Since

$$\begin{aligned} d\ln X &= \frac{1}{X}dX + \frac{1}{2}\sigma_X^2\left(-\frac{1}{X^2}\right)dt \\ &= \left(\frac{1}{X}\mu_X - \frac{1}{2}\sigma_X^2\frac{1}{X^2}\right)dt + \frac{\sigma_X}{X}dW \\ &\therefore \\ \mu &= \frac{1}{X}\mu_X - \frac{1}{2}\sigma_X^2\frac{1}{X^2} \\ \sigma &= \frac{\sigma_X}{X} \end{aligned}$$

i.e.,

$$\begin{aligned} \sigma_X &= \sigma X \\ \mu_X &= \left(\mu + \frac{1}{2}\sigma^2\right)X \end{aligned}$$

therefore,

$$d\left(\frac{f}{g}\right) = \left[-\frac{1}{2}\left(\sigma_f - \sigma_g\right)^2 + \frac{1}{2}\left(\sigma_f - \sigma_g\right)^2\right]dt + \left(\sigma_f - \sigma_g\right)\frac{f}{g}dW$$

i.e.,

$$d\left(\frac{f}{g}\right) = \left(\sigma_f - \sigma_g\right)\frac{f}{g}dW$$

in which there is no drift term, thus, $\frac{f}{g}$ is a martingale.

Note that

1. When the market price of risk $= \sigma_g$, it is a world of forward risk-neutral with respect to g.

2. Since $\frac{f}{g}$ is a martingale,

$$\begin{aligned} \therefore \quad \frac{f_0}{g_0} &= \mathbb{E}_g\left[\frac{f_T}{g_T}\right] \\ \Longrightarrow \\ f_0 &= g_0 \mathbb{E}_g\left[\frac{f_T}{g_T}\right] \end{aligned}$$

   (a) Money Market Account as the Numeraire:
   Since

$$dg = rgdt$$

   where r can be stochastic, but the volatility of $g = 0$, i.e., the market price of risk is zero for the money market. Then

$$f_0 = g_0 \mathbb{E}_{RN}\left[\frac{f_T}{g_T}\right]$$

   Since

$$\begin{aligned} g_0 &= 1 \\ g_T &= e^{\int_0^T r(\tau)d\tau} \end{aligned}$$

$$\therefore \quad f_0 = \mathbb{E}_{RN}\left[e^{-\int_0^T r(\tau)d\tau} f_T\right]$$

   If r is a constant, then

$$f_0 = e^{-rT}\mathbb{E}_{RN}\left[f_T\right]$$

   Hence, the money market account numerarie is equivalent to traditional risk-neutral world.

   (b) Zero-Coupon bond price as the Numeraire:
   Definition: $B(t,T)$ is the price at time t of a zero-coupon bond worth of \$1 at time T.
   Then,

$$\begin{aligned} g_T &= B(T,T) = 1 \\ g_0 &= B(0,T) \end{aligned}$$

$$\therefore \quad f_0 = B(0,T)\,\mathbb{E}_T\left[f_T\right]$$

   where $\mathbb{E}_T$ denotes the forward risk-neutral measure with respect to $B(t,T)$. Note that it is nice to have $B(0,T)$ outside $\mathbb{E}$-operator.
   Recall the forward price of f maturing at T is

$$F = \frac{f_0}{B(0,T)}$$

e.g., $F = S_0 e^{rT}$. Since

$$\begin{aligned} f_0 &= B(0,T)\,\mathbb{E}_T[f_T] \\ F &= \frac{f_0}{B(0,T)} \end{aligned}$$

$$\therefore F = \mathbb{E}_T[f_T]$$

i.e., In a forward risk-neutral measure with respect to $B(0,T)$, the forward price of f is equal to the expected future spot price. In contrast, in the traditional risk-neutral measure, futures price is equal to the expected future spot price.

Intuitive way (via binomial trees) of understanding change of numeraire:

Since

$$f_{now} = e^{-r\delta t}\left(qf_{up} + (1-q)\,f_{down}\right)$$

where q is the risk-neutral probability depending on the underlying movement. For another tradeable g, we have

$$g_{now} = e^{-r\delta t}\left(qg_{up} + (1-q)\,g_{down}\right)$$

therefore

$$\begin{aligned} \frac{f_{now}}{g_{now}} &= \frac{e^{-r\delta t}\left(qf_{up} + (1-q)\,f_{down}\right)}{e^{-r\delta t}\left(qg_{up} + (1-q)\,g_{down}\right)} \\ &= \frac{qg_{up}}{qg_{up} + (1-q)\,g_{down}}\frac{f_{up}}{g_{up}} + \frac{(1-q)\,g_{down}}{qg_{up} + (1-q)\,g_{down}}\frac{f_{down}}{g_{down}} \end{aligned}$$

If

$$q^* \equiv \frac{qg_{up}}{qg_{up} + (1-q)\,g_{down}}$$

Since

$$\begin{aligned} & \frac{(1-q)\,g_{down}}{qg_{up} + (1-q)\,g_{down}} + q^* \\ &= \frac{(1-q)\,g_{down}}{qg_{up} + (1-q)\,g_{down}} + \frac{qg_{up}}{qg_{up} + (1-q)\,g_{down}} \\ &= \frac{(1-q)\,g_{down} + qg_{up}}{qg_{up} + (1-q)\,g_{down}} = 1 \end{aligned}$$

i.e.,

$$\frac{(1-q)\,g_{down}}{qg_{up} + (1-q)\,g_{down}} = 1 - q^*$$

therefore,

$$\frac{f_{now}}{g_{now}} = q^*\left(\frac{f_{up}}{g_{up}}\right) + (1-q)\left(\frac{f_{down}}{g_{down}}\right)$$

Of course, q^* will vary from subtree to subtree. Therefore,

$$\frac{f_{now}}{g_{now}} = \mathbb{E}_* \left[\frac{f_{next}}{g_{next}} \right]$$

where $*$ denotes the expectation with respect to q^*. Iterate through the tree, we have

$$\frac{f(t)}{g(t)} = \mathbb{E}_* \left[\frac{f(T)}{g(T)} \right]$$

therefore,

$$\frac{f}{g} \text{ is a martingale with respect to } q^*$$

i.e.,

$$f(t) = g(t)\, \mathbb{E}_* \left[\frac{f(T)}{g(T)} \right]$$