Homework 4

1. Put $\Phi_n(x) := \prod_{(j,n)=1}(x - e^{\frac{2\pi j}{n}})$. Example: $\Phi_1(x) = x - 1$, $\Phi_2(x) = x + 1$.
   Show that
   - $\Phi_n \in \mathbb{Z}[x]$,
   - $\Phi_n(x)$ is irreducible over $\mathbb{Q}$,
   - $x^m - 1 = \prod_{d|m} \Phi_d(x)$,
   - $\Phi_m(x) = \prod_{d|m}(x^d - 1)^{\mu(m/d)}$.

2. Compute $c$, where
   $$\Phi_m(x) = x^{\phi(m)} + cx^{\phi(m)-1} + \cdots$$

3. Try to show that $3x^3 + 4y^3 + 5z^3 = 0$ is not solvable in $\mathbb{Z}$.

4. Show that
   $$\sum_{0 \leq j \leq n/3} \binom{n}{3j} = \frac{1}{3}(2^n + 2 \cos(\frac{\pi n}{3})).$$

5. Show that for all $m$, there are infinitely many primes $p$ such that $p = 1 \mod m$. 