Homework 7

1. Let $K = \mathbb{Q}(\sqrt{-5})$ and $a := (4 + \sqrt{-5}, 1 + 2\sqrt{-5}) \subset \mathcal{O}_K$. Show that $a$ is not a principal ideal and that it is a prime ideal.

2. Let $\alpha \in \mathcal{O}_K$ be an element such that $K = \mathbb{Q}(\alpha)$. Show that

$$\text{disc}(\alpha) = \text{disc}(K/\mathbb{Q}) \cdot t^2,$$

for some $t \in \mathbb{Z}$. This $t$ is called the \textit{index} of $\alpha$.

3. Show that if $\alpha \in \mathcal{O}_K$ has index one then $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

4. Given coprime integral ideals $a, b \subset \mathcal{O}_K$ and $\alpha, \beta \in \mathcal{O}_K$ show that there exists a $\lambda \in \mathcal{O}_K$ such that

$$\lambda \equiv \alpha \pmod{a}, \quad \lambda \equiv \beta \pmod{b}.$$

5. Let $K = \mathbb{Q}(\zeta_7)$. Find a unit of infinite order.