Homework Set 10

1. Let $X$ and $Y$ be arbitrary non-empty sets and $f : X \to Y$ be an arbitrary function. For any $E \subset Y$, $f^{-1}(E) = \{x \in X : f(x) \in E\}$ denotes the pre-image of $E$ in $X$. Show that

(a) $f(A \cup B) = f(A) \cup f(B)$, but in general $f(A \cap B) \subset f(A) \cap f(B)$. Give an example that shows strict inclusion can happen in the latter.

(b) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ and $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ for all $C \subset Y$ and $D \subset Y$. Compare with (a).

(c) $f^{-1}(E^c) = f^{-1}(E)^c$ for all $E \subset Y$. Show that, in general, there need not be any inclusion relation between $f(A^c)$ and $f(A)^c$.

2. Again let $X$ and $Y$ be arbitrary non-empty sets and $f : X \to Y$ be an arbitrary function. Show that

(a) $f^{-1}(f(A)) \supset A$ for all $A \subset X$ and $f(f^{-1}(B)) \subset B$ for all $B \subset Y$.

(b) $f$ is 1–1 if and only if $f^{-1}(f(A)) = A$ for all $A \subset X$.

(c) $f$ is onto if and only if $f(f^{-1}(B)) = B$ for all $B \subset Y$.

3. Let $X$ and $Y$ be metric spaces and $f : X \to Y$ be a continuous function. Recall that $\overline{A}$ denotes the closure of $A$.

(a) Show that $f(\overline{E}) \subset \overline{f(E)}$ for all $E \subset X$.

(Hint: $y \in f(\overline{E})$ implies there is a sequence $(x_n)$ in $E$ such that $y = f(\lim x_n)$.)

(b) Show that for all $y \in Y$ the set $E_y = \{x \in X : f(x) = y\}$ is a closed set in $X$. In particular, show that the set of points on which a continuous real-valued function vanishes (i.e., takes the value zero) is a closed set. (Hint: Obviously $E_y = f^{-1}\{\{y\}\}$.)

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that $f$ is continuous at $x = 0$ and discontinuous everywhere else.