A REMARK ON THE EXISTENCE OF SUITABLE VECTOR FIELDS RELATED TO THE DYNAMICS OF SCALAR SEMILINEAR PARABOLIC EQUATIONS

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Let Ω be a bounded smooth domain in \( \mathbb{R}^n \), \( \nu \) be the outer normal direction of \( \partial \Omega \) and \( f \in C^\infty (\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \). The infinite dimensional dynamical system defined by

\[
\begin{cases}
  u_t = \Delta u + f(x, u, \nabla u), & x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

on suitable Sobolev spaces has attracted much interest (see \([A, H]\) and more recent references at the end of this note). Many efforts have been made to show the complexity of its dynamical behavior (see some survey papers and recent articles \([DP, P1, P2, P3, Pr, PR, R]\) and the references therein). In particular, the following nice result was proven in \([P2]\): if there exists a smooth vector field \( \Phi \) on \( \overline{\Omega} \), \( \Phi = (\phi_1, \cdots, \phi_n) \) such that

\[
\text{rank} (\Phi(x), \partial_1 \Phi(x), \cdots, \partial_n \Phi(x)) = n \text{ for all } x \in \overline{\Omega},
\]

then for any smooth vector field \( X \) on \( \mathbb{R}^n \), there exists a smooth function \( f \), such that \( \text{span} \{\phi_1, \cdots, \phi_n\} \) is invariant under (1) and for any integral curve of \( X \), \( c = c(t), u = \sum_{i=1}^n c_i(t) \phi_i(x) \) is a solution to (1). Moreover, it was shown that such kind of vector field always exists on a starshaped domain. The main result of this short note is a classification of all the domains on which one may find this type of vector fields. More precisely, we have

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be an open bounded smooth domain, then the necessary and sufficient condition for the existence of a smooth map \( F : \overline{\Omega} \to \mathbb{R}^n \) with

\[
\begin{cases}
  \text{rank} (F(x), \partial_1 F(x), \cdots, \partial_n F(x)) = n & \text{for any } x \in \overline{\Omega}, \\
  \frac{\partial F}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

is that \( \overline{\Omega} \) is diffeomorphic to \( \overline{B}_1 \) or \( \overline{B}_2 \setminus B_1 \).

**Remark 1.** In fact, if \( \overline{\Omega} \) is diffeomorphic to \( \overline{B}_1 \), then any solution to (2), \( F \), must have exactly one zero in \( \Omega \). If \( \overline{\Omega} \) is diffeomorphic to \( \overline{B}_2 \setminus B_1 \), then any solution to (2), \( F \), does not vanish at all. These conclusions will follow from the arguments below.

First we reduce the existence of such a vector field to the existence of vector field with less restrictions.

**Lemma 1.** Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be an open bounded smooth domain, if there exists a smooth map \( G : \overline{\Omega} \to \mathbb{R}^n \) such that

\[
\text{rank} (G(x), \partial_1 G(x), \cdots, \partial_n G(x)) = n \text{ for any } x \in \overline{\Omega}
\]
then it is clear

**Corollary 1.** Assume \( \Omega_1 \) is diffeomorphic to \( \Omega_2 \), and for \( \Omega_1 \) we may find a solution to (2), then we may find a solution to (2) for \( \Omega_2 \) too.

To derive the necessary condition for the existence of a vector field satisfying (2), we will need

**Lemma 2.** Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be an open bounded smooth domain, if there exists a smooth map \( H : \overline{\Omega} \to S^{n-1} \) such that

\[
\text{rank}(\partial_1 H(x), \ldots, \partial_n H(x)) = n - 1 \quad \text{for any } x \in \overline{\Omega}
\]

and

\[
\dim \text{im} (H|_{\partial \Omega})_{*,x} = n - 1 \quad \text{for any } x \in \partial \Omega,
\]

then \( \overline{\Omega} \) is diffeomorphic to \( \overline{B_2 \setminus B_1} \).

**Proof.** First we claim that each path connected component of \( \partial \Omega \) is diffeomorphic to \( S^{n-1} \). This is clear when \( n = 2 \). If \( n \geq 3 \), since \( H|_{\partial \Omega} \) has full rank everywhere and \( \partial \Omega \) is compact, \( H|_{\partial \Omega} : \partial \Omega \to S^{n-1} \) is a covering map (see [M]). Since \( S^{n-1} \) is simply connected, we see each path connected components of \( \partial \Omega \) must be diffeomorphic to \( S^{n-1} \). Indeed, the restriction of \( H \) to such a component serves as a diffeomorphism.

To proceed, we observe that from the assumption on \( H \), it follows from implicit function theorem that for any \( \xi \in S^{n-1} \), \( H^{-1}(\xi) \) is a smooth one dimensional submanifold of \( \overline{\Omega} \), moreover \( H : \overline{\Omega} \to S^{n-1} \) is a smooth fiber bundle (see [M]). Fix
a point \( x_0 \in \partial \Omega \), let \( \xi_0 = H(x_0) \) and \( \Gamma = H^{-1}(\xi_0) \), then we have an exact sequence (see Theorem 6.7 of chapter VII in [B])

\[
\pi_{n-1}(\Gamma, x_0) \rightarrow \pi_{n-1}(\overline{\Omega}, x_0) \rightarrow \pi_{n-1}(S^{n-1}, \xi_0) \rightarrow \pi_{n-2}(\Gamma, x_0).
\]

If \( n \geq 3 \), then both \( \pi_{n-1}(\Gamma, x_0) \) and \( \pi_{n-2}(\Gamma, x_0) \) vanishes, this shows \( \pi_{n-1}(\overline{\Omega}, x_0) \cong \mathbb{Z} \) and hence \( \overline{\Omega} \) is diffeomorphic to \( \overline{B}_2 \setminus B_1 \). If \( n = 2 \), then since \( \pi_1(\Gamma, x_0) \) vanishes and \( \pi_0(\Gamma, x_0) \) is finite, we see \( \pi_1(\overline{\Omega}, x_0) \) is again isomorphic to \( \mathbb{Z} \), this shows \( \overline{\Omega} \) must be diffeomorphic to \( \overline{B}_2 \setminus B_1 \).

Now we are ready to prove the theorem.

**Proof of theorem 1.** First if \( \Omega = B_1 \) or \( B_2 \setminus B_1 \), then \( G(x) = x \) satisfies the assumption in the Lemma 1, half of the theorem follows from the lemma and Corollary 1. On the other hand, assume for some \( \Omega \), we may find a smooth map \( F \) satisfying (2).

For \( x \in \partial \Omega \), choose a base for the tangent space of \( \partial \Omega \) at \( x \), namely \( e_1, \cdots, e_{n-1} \), then

\[
\text{rank } (F(x), \partial_1 F(x), \cdots, \partial_n F(x)) = \text{rank } (F(x), F_* e_1, \cdots, F_* e_{n-1}, F_* \nu) = \text{rank } (F(x), F_* e_1, \cdots, F_* e_{n-1}) = n,
\]
we see \( F(x) \neq 0 \) on \( \partial \Omega \). Moreover, it follows from the fact

\[
\text{rank } (F(x), \partial_1 F(x), \cdots, \partial_n F(x)) = n \text{ for any } x \in \overline{\Omega}
\]
that the zeroes of \( F \) in \( \Omega \) must be isolated, hence only finitely many, say \( x_1, \cdots, x_m \). Then for \( \varepsilon > 0 \) small enough, let

\[
U = \Omega \setminus \bigcup_{i=1}^m \overline{B}_\varepsilon(x_i),
\]
we know

\[
\text{rank } (F(x), \partial_1 F(x), \cdots, \partial_n F(x)) = n \text{ for any } x \in U
\]
and

\[
\text{dim span } \left\{ F(x), \text{im } (F|_{\partial U})*x \right\} = n \text{ for any } x \in \partial U.
\]

Let

\[
H(x) = \frac{F(x)}{|F(x)|} \text{ for } x \in U,
\]
then clearly

\[
\text{rank } (\partial_1 H(x), \cdots, \partial_n H(x)) = n - 1 \text{ for any } x \in U
\]
and

\[
\text{dim im } (H|_{\partial U})*x = n - 1 \text{ for any } x \in \partial U.
\]
It follows from the Lemma 2 that \( U \) must be diffeomorphic to \( \overline{B}_2 \setminus B_1 \), hence \( \overline{\Omega} \) must be diffeomorphic to either \( \overline{B}_1 \) or \( \overline{B}_2 \setminus B_1 \).

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