1. **Consider the set** \( X = \{1, 2, 3\} \).

   (a) **List all invertible functions from** \( X \) **to** \( X \).
   
   Solution: I will list their ordered pairs:
   
   \[
   \{(1, 1), (2, 2)(3, 3)\}, \{(1, 2), (2, 3), (3, 1)\}, \{(1, 3), (2, 1), (3, 2)\}, \\
   \{(1, 1), (2, 3), (3, 2)\}, \{(1, 3), (2, 2), (3, 1)\}, \{(1, 2), (2, 1), (3, 3)\}.
   \]

   (b) **List all invertible functions from** \( X \) **to the set** \( \{1, 2, 3, 4\} \).
   
   Solution: There are none.

   (c) **List all equivalence relations on** \( X \) **and then list all partitions of** \( X \).
   
   Solution: The equivalence relations on \( X \) are
   
   \[
   \{(1, 1), (2, 2), (3, 3)\}, \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}, \\
   \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}, \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}, \\
   \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}.
   \]

   The partitions of \( X \) are (I will not include the empty set in any of my partitions, although you can)
   
   \[
   \{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{3\}, \{2\}\}, \\
   \{\{1\}, \{2, 3\}\}, \{\{1, 2\}, \{3\}\}.
   \]

   (d) **List all relations which are both reflexive and symmetric.**
   
   Solution:
   
   \[
   \{(1, 1), (2, 2), (3, 3)\}, \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}, \\
   \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}, \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}, \\
   \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}, \\
   \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}, \\
   \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 3), (3, 2)\}, \\
   \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2), (1, 3), (3, 1)\}.
   \]
2. Out of 40 students, 30 do not have American citizenship and 10 have Canadian citizenship. Five of the students who have American citizenship do not have Canadian citizenship. How many have neither Canadian nor American citizenship?

Solution: Let $A$ be the set of students who have American citizenship, let $C$ be the set of students who have Canadian citizenship, and let $S$ be the set of all students. We are given that

$$ |S| = 40, |S \setminus A| = 30, |C| = 10, \text{ and } |A \setminus C| = 5. $$

Since we know

$$ A \cup C = (A \setminus C) \cup C \text{ and } (A \cup C) \cap C = \{\}, $$

this gives

$$ |A \cup C| = |A \setminus C| + |C| = 5 + 10 = 15. $$

This is the number of students with American or Canadian citizenship. Therefore, the number of students with neither American nor Canadian citizenship is

$$ |(S \setminus A) \cap (S \setminus C)| = |S| - |A \cup C| = 40 - 15 = 25. $$

3. (a) Solve the following recurrence relation, writing the term $L(n)$ as a function of $n$.

$$
\begin{align*}
L(n) &= 4L(n-1) - 4L(n-2) \\
L(0) &= 1 \\
L(1) &= 3
\end{align*}
$$

Solution: The characteristic equation is

$$ r^2 - 4r + 4 = 0 $$

$$(r - 2)^2 = 0, $$

which has a double root at 2. In other words,

$$ r = r_1 = r_2 = 2. $$

Therefore the general solution is

$$ L(n) = c_1(2)^n + c_2n(2)^n. $$

To solve for the constants, use the initial conditions:
\[
\begin{aligned}
1 &= L(0) = c_1(2)^0 + c_2(0)(2)^0 \\
3 &= L(1) = c_1(2)^1 + c_2(1)(2)^1 \\
1 &= c_1 \\
3 &= 2c_1 + 2c_2
\end{aligned}
\]

This gives \( c_1 = 1 \) and \( c_2 = 1/2 \). Therefore the specific solution is

\[
L(n) = 2^n + (1/2)n2^n
\]

\[
L(n) = 2^n + n2^{n-1}.
\]

(b) I have an algorithm which takes as input a value \( n \geq 2 \) and which executes \( 1000n^2 + 200n - 40 \) lines and you have an algorithm which accomplishes the same task but which executes \( n^3 - 2n \) lines. What is the run-time complexity of each? Prove your answer. Which do you think is better for large values of \( n \)?

Solution: The first algorithm has run-time complexity \( \Theta(n^2) \). To see this,

\[
1000n^2 + 200n - 40 \leq 1000n^2 + 200n \leq 1000n^2 + 200n^2 = 1200n^2,
\]

so that it is \( O(n^2) \). Further,

\[
1000n^2 + 200n - 40 \geq 1000n^2 - 40 \geq 1000n^2 - 40n^2 = 960n^2,
\]

so that it is \( \Omega(n^2) \). For the second algorithm, we have run-time complexity \( \Theta(n^3) \). First,

\[
n^3 - 2n \leq n^3,
\]

giving that it is \( O(n^3) \). Last, since for \( n \geq 2 \),

\[
\begin{aligned}
n^2 &\geq 4 \\
n^3 &\geq 4n \\
(1/2)n^3 &\geq 2n \\
-2n &\geq -(1/2)n^3,
\end{aligned}
\]

we get

\[
n^3 - 2n \geq n^3 - (1/2)n^3 = (1/2)n^3.
\]
This means the second algorithm has run-time complexity which is $\Omega(n^3)$. Therefore it is $\theta(n^3)$. For large values of $n$, the second algorithm will take much more time than the first (we may have to have a very large value of $n$ to actually see this though.) Therefore we are justified in saying that the first algorithm is better than the second.

4. (a) **Does the first graph on the back have an Euler cycle? Why or why not? If so, can you find one?**

Solution: There is an Euler cycle because the degree of each vertex is even. One possible Euler cycle is given by traveling the edges in the following order:

274638234567812.

(b) **Is there a Hamiltonian cycle in the second graph on the back?**

Solution: There is no Hamiltonian cycle. The reason is the following. If there were such a cycle, then the edge with weight 4 would have to be in it. Supposing that we start traveling on this cycle on the left side of the graph, then after we cross this edge to the right side, we can never come back to the left side, because this would entail traversing this edge again.

(c) **List the order in which edges are added to a growing tree given by Prim’s algorithm applied to the second graph on the back. Check that the resulting tree is, in fact, a minimal spanning tree by finding all possible spanning trees and calculating their total weights.**

Solution: The order in which edges are added by Prim’s algorithm is 1, 2, 3, 4, 6, 7, 8. To check this is a minimal spanning tree, we need to find all spanning trees. Notice that each spanning tree will contain the edge of weight 4. To construct a spanning tree, we need only choose three edges from the left side and three edges from the right side. There are sixteen ways to do that:
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