Louville Equations

1. Truncated Burgers equation.

The inviscid Burger’s equation is
\[
\frac{\partial}{\partial t} u + \frac{1}{2} (u^2)_x = 0
\]

Let \( P_\Lambda f = f_\Lambda \) denote the finite Fourier series truncation, i.e.,
\[
P_\Lambda f = f_\Lambda = \sum_{|k| \leq \Lambda} \hat{f}_k e^{ikx}
\]

Here, all functions, such as, \( u, f \) are assumed to be \( 2\pi \)-periodic.

We define the Fourier-truncated Burgers equation as
\[
\frac{\partial}{\partial t} u_\Lambda + \frac{1}{2} P_\Lambda (u^2_\Lambda)_x = 0
\]

and note that
\[
\begin{align*}
u_\Lambda (t) &= \sum_{|k| \leq \Lambda} \hat{u}_k (t) e^{ikx} \\
\hat{u}_{-k} &= \hat{u}_k^*
\end{align*}
\]

Show that the truncated Burgers equation has a Liouville property in the variables \( \{ \hat{u}_k \}, 1 \leq |k| \leq \Lambda \). (more precisely, in \( a_k, b_k \), where \( \hat{u}_k = a_k + ib_k \)).

Kinetic Theory:
Properties of Collision Operator:

1. Starting from the Newton second law for a central force \( F = -\nabla U (|x_2 - x_1|) \) between two bodies:
\[
\begin{align*}
\frac{dx}{dt} &= v, \quad \frac{dx_1}{dt} = v_1, \\
m \frac{dv}{dt} &= - \frac{\partial U}{\partial \rho}, \quad m \frac{dv_1}{dt} = \frac{\partial U}{\partial \rho}
\end{align*}
\]

where \( \rho = x_2 - x_1 \), show that
\[
\begin{align*}
v' &= v + \dot{\alpha} (\dot{\alpha} \cdot V) \\
v'_1 &= v_1 - \dot{\alpha} (\dot{\alpha} \cdot V)
\end{align*}
\]

and
\[
\begin{align*}
v &= v' + \dot{\alpha} (\dot{\alpha} \cdot V') \\
v_1 &= v'_1 - \dot{\alpha} (\dot{\alpha} \cdot V')
\end{align*}
\]

Hence
\[
V' = V - 2\dot{\alpha} (\dot{\alpha} \cdot V)
\]

where the prime indicates that the variables denote the quantities before the collision, \( V \equiv v_1 - v \), \( V' \equiv v'_1 - v' \) are the relative velocities, and \( \dot{\alpha} \) is the unit vector which bisects the angle between the vectors \( V \) and \( -V' \). (see figure)
2. Assuming that the two-body potential is completely repulsive, then the angle \( \theta = \theta(r) \) is an increasing function of \( r \), the impact parameter.

Define

\[
B(\theta, V) = |V| r \frac{\partial r}{\partial \theta}
\]

show that the Boltzmann equation can be written in the form

\[
\frac{\partial}{\partial t} f + v \cdot \frac{\partial}{\partial x} f = \frac{1}{m} \int_{v \in \mathbb{R}^3} d^3v_1 \int_0^{2\pi} d\eta \int_0^{\pi/2} d\theta (f' f_1 - f f_1) B(\theta, V)
\]

(3)

For example, if the interparticle potential is hard-core:

\[
U = \begin{cases}
0, & r \geq \sigma \\
\infty, & r \leq \sigma
\end{cases}
\]

then,

\[
r = \sigma \sin \theta \\
B(\theta, V) = \frac{|V|}{\sigma^2} \cos \theta \sin \theta.
\]

3. Define bilinear integral operator:

\[
Q(f, g) = \frac{1}{2m} \int (f' g_1' + f_1' g' - f g_1 - f_1 g) B(\theta, V) d^3v_1 d\eta d\theta
\]

for any functions \( f \) and \( g \). Obviously, \( Q(f, g) = Q(g, f) \), \( Q(f, g) = Q(f, f) \), i.e., the collision integral when \( f = g \). Consider the eight-fold integral:

\[
\int_{v \in \mathbb{R}^3} d^3v Q(f, g) \phi(v) = \frac{1}{2m} \int \phi(v) (f' g_1' + f_1' g' - f g_1 - f_1 g) B(\theta, V) d^3v_1 d^3v_1 d\eta d\theta
\]

for any function \( \phi(v) \). Using Eqs (1) and (2), show that

\[
\int_{v \in \mathbb{R}^3} d^3v Q(f, g) \phi(v) = \frac{1}{8m} \int (\phi + \phi_1 - \phi' - \phi'_1) (f' g_1' + f_1' g' - f g_1 - f_1 g) B(\theta, V) d^3v_1 d^3v_1 d\eta d\theta
\]
where $\phi = \phi(v)$, $\phi_1 = \phi(v_1)$, $\phi' = \phi(v')$, $\phi'_1 = \phi(v'_1)$. If $f = g$, then

$$\int_{v \in \mathbb{R}^3} d^3v Q(f, f) \phi(v) = \frac{1}{4m} \int (\phi + \phi_1 - \phi' - \phi'_1) (f' f'_1 - f f_1) B(\theta, V) d^3v_1 d^3v d\theta.$$  

Therefore if $\phi + \phi_1 = \phi' + \phi'_1$ holds almost everywhere, then

$$\int_{v \in \mathbb{R}^3} d^3v Q(f, g) \phi(v) = 0$$

independent of the functions $f, g$. Thus, $\phi$ is referred to as the collision invariant if $\phi + \phi_1 = \phi' + \phi'_1$.

4. Show that for any $f \geq 0$,

$$\int (\log f) Q(f, f) d^3v \leq 0$$

the equality holds iff $f(v) = e^{a+b \cdot v + c |v|^2}$ (Hint: $(x - y) \log \frac{x}{y} \geq 0$, and note that $B(\theta, V) \geq 0$).

H-Theorem:

1. Show that H-theorem holds for the BGK equation:

$$\left( \frac{\partial}{\partial t} + v \cdot \nabla_x \right) f(x, v, t) = -\frac{f - f^{(0)}}{\tau}$$

where

$$f^{(0)}(x, v, t) = \rho(x, t) \left( \frac{m}{2\pi k_B T(x, t)} \right)^{3/2} e^{-\frac{m}{2k_BT} (v - u(x, t))^2}$$

with the assumption that the local equilibrium hydrodynamic fields are identical to the true fields, i.e.,

$$\langle A \rangle = \frac{\int d^3p A f}{\int d^3p f} \quad \text{is replaced with} \quad \langle A \rangle_0 = \frac{\int d^3p A f^{(0)}}{\int d^3p f^{(0)}}, \quad (4)$$

In the proof, you may need

$$\int d^3v f \log f^{(0)} = \int d^3v f^{(0)} \log f^{(0)}$$

which is a special case of the assumption $(4)$.

2. Show that the H-theorem holds for the Boltzmann equation (3) for an isolated system.

Transport phenomena:

1. In our derivation of Navier-Stokes equation, we used a relaxation approximation. To analyze the constitutive relationships, we used

$$\phi = -\tau \left( \frac{\partial}{\partial t} + v \cdot \nabla_x + \frac{F}{m} \cdot \nabla_v \right) f^{(0)}$$

$$= -\tau f^{(0)} \left[ \frac{1}{\rho} D(\rho) + \frac{1}{\theta} \left( \frac{m}{2\theta} U^2 - \frac{3}{2} \right) D(\theta) + \frac{m}{\theta} U_j D(u_j) - \frac{1}{\theta} F \cdot U \right]. \quad (5)$$

Derive this expression as outlined in the notes.
Large Deviations:

1. Estimate the probability that a stamp (mass = 0.1g) resting on a desk top at room temperature (300K) will spontaneously fly up to a height of $10^{-8}$ cm above the desk top. (Hint: Imagine there is an infinite number of noninteracting stamps placed side by side. Formulate an argument showing that these stamps obey the Maxwell-type distribution.)

2. For Curie-Weiss model, i.e.

$$ E\{s_i\} = -\frac{J}{2N} \sum_{i,j} s_i s_j - H \sum_i s_i, $$

instead of using the large deviation principle, show that

(a) 

$$ \lim_{N \to \infty} \frac{1}{N} \log Z_N(\beta, H) = \sup_{m \in \mathbb{R}} \left\{ \frac{1}{2} \beta J m^2 + \beta H m - I_\rho(m) \right\} $$

where

$$ I_\rho(m) = \frac{1}{2} (1 - m) \log(1 - m) + \frac{1}{2} (1 + m) \log(1 + m) $$

for $|m| \leq 1$ and $I_\rho(m) = \infty$ for $|m| > 1$.

(b) If $m^*$ attains the supremum in Eq.(6), then

$$ m^* = \tanh[\beta(Jm^* + H)]. $$

Hint: Note that

$$ \exp\left(\frac{1}{2} y^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2} y^2 - \frac{1}{2} t^2} dt $$

and prove that

$$ \lim_{N \to \infty} \frac{1}{N} \log Z_N(\beta, H) = \sup_{t \in \mathbb{R}} \left\{ \log \cosh t - \frac{(t - \beta H)^2}{2\beta J} \right\} $$

then show that

$$ \sup_{t \in \mathbb{R}} \left\{ \log \cosh t - \frac{(t - \beta H)^2}{2\beta J} \right\} = \sup_{m \in \mathbb{R}} \left\{ \frac{1}{2} \beta J m^2 + \beta H m - I_\rho(m) \right\} $$

Mean-Field:

1. If the Hamiltonian is

$$ \mathcal{H} = \frac{J}{N} S^2 + HS, $$

$$ S = \sum_{i=1}^{N} s_i, $$

where $H$ is the magnetic field. Show that for this model mean-field theory is exact in the limit $N \to \infty$, by the following steps:

(a) Write down the partition function, and express it as an integral over an auxiliary field $\phi$. The configuration sum can then be performed. Hint: Note that

$$ \exp\left(\frac{1}{2} y^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2} y^2 - \frac{1}{2} t^2} dt $$

(b) Use the saddle-point method to evaluate the partition function in the limit $N \to \infty$. Show that the saddle-point condition gives the mean-field theory.