1 Discrete-Time Hedging

1. Discrete-time hedging without transaction costs;
2. Discrete-time hedging with transaction costs.

1.1 Discrete-time Hedging with Zero Transaction Costs

\[
\frac{dS}{S} = \left( \mu + \frac{1}{2} \sigma^2 \right) \delta t + \sigma Z \sqrt{\delta t}, \quad Z \sim \mathcal{N}(0, 1)
\]

For a small (non-infinitesimal) \( \delta t \),

\[
\frac{\delta S}{S} = \left( \mu + \frac{1}{2} \sigma^2 \right) \delta t + \sigma Z \sqrt{\delta t} + O(\delta t^{3/2}) \tag{1}
\]

which can be heuristically seen from

\[
S(t + \delta t) = S(t) e^{\mu \delta t + \sigma(W(t + \delta t) - W(t))}
\]

\[
\frac{\delta S}{S} = \frac{S(t + \delta t) - S(t)}{S(t)} = e^{\mu \delta t + \sigma(W(t + \delta t) - W(t))} - 1
\]

\[
= \mu \delta t + \sigma (W(t + \delta t) - W(t)) + \frac{1}{2} \sigma^2 (W(t + \delta t) - W(t))^2 + O(\delta t^{3/2})
\]

\[
= \left( \mu + \frac{1}{2} \sigma^2 \right) \delta t + \sigma Z \sqrt{\delta t} + O(\delta t^{3/2})
\]

Here, we note that \( E[(W(t + \delta t) - W(t))^2] = \delta t \) and \( Var[(W(t + \delta t) - W(t))^2] = 2(\delta t)^2 \).

Recall: The replicating portfolio we used in deriving the Black-Scholes PDE is:

- \( \Delta \) shares of stock;
- \( \Pi \) dollars of the risk-free security

and

\[
V = \Pi + \Delta S
\]

or

\[
\Pi = V - \Delta S
\]

As \( \delta t \to 0 \), we obtain

\[
\frac{1}{2} \sigma^2 S^2 V_{SS} + V_t - r(V - V_S S) = 0
\]
Now view $\delta t$ as the **revision interval**
Over $\delta t$, the return of this portfolio is

$$
\delta P = \Delta \delta S + r \Pi \delta t + O(\delta t^2)
\approx \text{approx. to continuous compounding}
$$

$$
= \Delta S \left( \frac{\delta S}{S} \right) + r \Pi \delta t
$$

and over this interval $\delta t$, the change of the value of $V$ is

$$
\delta V = V_S \delta S + V_t \delta t + \frac{1}{2} V_{SS} (\delta S)^2 + O(\delta t^{3/2})
\quad \because \text{Taylor Expansion and Eq. (1)}
\quad = V_S S \left( \frac{\delta S}{S} \right) + V_t \delta t + \frac{1}{2} V_{SS} S^2 \left( \frac{\delta S}{S} \right)^2 + O(\delta t^{3/2})
$$

The difference between the hedging (replicating) portfolio and the value of the option is

$$
\delta H = \delta P - \delta V
\quad = (\Delta S - V_S S) \left( \frac{\delta S}{S} \right) + (r \Pi - V_t) \delta t - \frac{1}{2} V_{SS} S^2 \left( \frac{\delta S}{S} \right)^2 + O(\delta t^{3/2}) \quad (2)
$$

Note that, in our derivation of the Black-Scholes PDE, we assume replicating portfolio at $0, \delta t, 2\delta t, \cdots, T$

$$
\Delta = V_S \quad (3)
\Pi = V - V_S S \quad (4)
$$

Therefore,

1.

$$
P = \Delta S + \Pi = V \quad \text{at each } t = k\delta t
\quad P = f(S_T) \quad \text{at } t = T
$$

i.e., replicating.

2. The portfolio is not self-financing, we will use $\delta H \neq 0 \implies \delta P \neq \delta V$
i.e., not self-financing. Hence, $\delta H$ is an additional contribution needed over $(t, t + \delta t)$. How can we see that $\delta H \neq 0$? This can be shown as follows: Plugging Eqs. (3) and (4) into Eq. (2) leads to

$$
\delta H = 0 \cdot \frac{\delta S}{S} + [r (V - V_S S) - V_t] \delta t - \frac{1}{2} V_{SS S}^2 \left( \frac{\delta S}{S} \right)^2 + O (\delta t^{3/2})
$$

Now note that due to the BS PDE:

$$
r (V - V_S S) - V_t = \frac{1}{2} \sigma^2 V_{SS S}^2
$$

(Here, the BS PDE can be thought of as obtained via, say, binomial model, where hedging is perfect, or from the Black-Scholes formula, which satisfies the BS PDE.) Therefore,

$$
\delta H = \frac{1}{2} V_{SS S}^2 \left[ \sigma^2 \delta t - \left( \frac{\delta S}{S} \right)^2 \right] + O (\delta t^{3/2})
$$

Note that

(a) 

$$
V_{SS S}^2 \sim O (1).
$$

(b) The term $(\mu + \frac{1}{2} \sigma^2) \delta t$ in $\frac{\delta S}{S}$ will contribute terms of order higher than $\delta t$ in $\left( \frac{\delta S}{S} \right)^2$, therefore

$$
\sigma^2 \delta t - \left( \frac{\delta S}{S} \right)^2 = \sigma^2 \delta t - \sigma^2 Z^2 \delta t + O (\delta t^{3/2}) = \sigma^2 (1 - Z^2) \delta t
$$

Hence,

$$
\delta H = \frac{1}{2} \sigma^2 S^2 V_{SS S}^2 (1 - Z^2) \delta t
$$

This is Gamma, i.e., we have $\Delta$-hedged, but with $\Gamma$-exposure therefore,

$$
\delta H \sim O (\delta t)
$$

But, neglecting terms of $O (\delta t^{3/2})$

$$
\mathbb{E} [\delta H] = \frac{1}{2} \sigma^2 S^2 V_{SS S} \mathbb{E} [1 - Z^2] \delta t = 0
$$
and
\[ \text{Var} [\delta H] \sim O(\delta t)^2 \]
therefore, \( \delta H \) is a random variable with zero mean and variance of order \( O(\delta t^2) \).

But the actual cost, i.e., the total hedging error over the entire interval \([0, T]\) is not large. Why? Let us explain:

Let
\[ n = \frac{T}{\delta t} \]
which is the number of revision intervals.

The total hedge error
\[ = \sum_{i=0,\delta t,2\delta t, \ldots, T-\delta t} \delta H_i \]
\[ = \sum_{i=1}^{n} \frac{1}{2} \sigma^2 S^2(t_i) \frac{\partial^2 V}{\partial S^2}(S(t_i), t_i) (1 - Z_i^2) \delta t \]
whose mean
\[ = 0 \]
variance
\[ = O\left(n(\delta t)^2\right) \]
\[ = O\left(\frac{T}{\delta t}(\delta t)^2\right) \]
\[ = O(T\delta t) \rightarrow 0 \quad \text{as } \delta t \rightarrow 0 \]
i.e.,
\[ \text{the hedging error} \rightarrow 0 \quad \text{as } \delta t \rightarrow 0 \]

Note that, for a binomial tree, we have a perfect hedging at every period. As \( \delta t \rightarrow 0 \), we obtain the BS formula. Therefore, Ito calculus setting and binomial tree results converge on the continuous time hedging, which requires infinitely many revisions.

Question:
If there is a transaction cost associated with each hedging revision, will infinitely many revisions lead to infinite transaction cost in total?

1.2 Discrete-time Hedge with Transaction Costs (TC)

cf. Wilmott et al, Chap 16

Assumptions:
1. Transaction cost is proportional to the monetary value of the transaction, e.g., if \( \nu \) shares are bought, \( \nu > 0 \) at price \( S \), (if \( \nu \) shares are sold, \( \nu < 0 \)) then, the transaction costs are

\[
\kappa |\nu| S
\]

where \( \kappa \) is a constant, depending on type of investors, e.g., \( \kappa \) is small usually for banks.

2. The hedged portfolio has expected return equal to that of risk-free security (this is a brave instance that we have gone beyond no-arbitrage, risk-neutral dogma!).

Note that, in the discrete-time hedge for the case of no transaction costs, the hedge is carried out in the same sense, i.e., the expected error vanishes as \( \delta t \to 0 \).

We have

\[
\delta S = \left( \mu + \frac{1}{2} \sigma^2 \right) S \delta t + \sigma S \sqrt{\delta t} Z + O(\delta t^{3/2})
\]

\[
\Pi = V - \Delta S
\]

1. Without transaction cost:

\[
\delta \Pi = \delta V - \Delta \delta S
\]

\[
= \frac{\partial V}{\partial S} \delta S + \frac{\partial V}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (\delta S)^2 - \Delta \delta S + O(\delta t^{3/2})
\]

\[
= \left( \frac{\partial V}{\partial S} - \Delta \right) \left[ \bar{\mu} S \delta t + \sigma S \sqrt{\delta t} Z \right]
\]

\[
+ \frac{\partial V}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \left( \bar{\mu} S \delta t + \sigma S \sqrt{\delta t} Z \right)^2 + O(\delta t^{3/2})
\]

\[
= \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) Z \sqrt{\delta t}
\]

\[
+ \left[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \bar{\mu} \left( \frac{\partial V}{\partial S} - \Delta \right) S + \frac{\partial V}{\partial t} \right] \delta t
\]

2. With transaction costs:

\[
\delta \Pi \to \delta \Pi = \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) Z \sqrt{\delta t}
\]

\[
+ \left[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \bar{\mu} \left( \frac{\partial V}{\partial S} - \Delta \right) S + \frac{\partial V}{\partial t} \right] \delta t
\]

\[- \kappa S |\nu|
\]

Transaction costs, which always reduce the value of portfolio.
Hedging Strategy:

1. At time $t$:
   \[ \Delta (t) = \frac{\partial V}{\partial S} (S,t) \]
   which is the number of shares of stock held at time $t$.

2. After $\delta t$, rehedging:
   \[ \Delta (t + \delta t) = \frac{\partial V}{\partial S} (S + \delta S, t + \delta t) \]
   \[ \uparrow \]
   new stock price

Thus, the number of shares we have to trade to maintain a hedged position is

\[ \nu = \Delta (t + \delta t) - \Delta (t) \]
\[ = \frac{\partial V}{\partial S} (S + \delta S, t + \delta t) - \frac{\partial V}{\partial S} (S, t) \]

Taylor expansion:

\[ \frac{\partial V}{\partial S} (S + \delta S, t + \delta t) = \frac{\partial V}{\partial S} (S, t) \]
\[ + \frac{\partial^2 V}{\partial S^2} (S, t) \delta S \] dominate term
\[ (\because \delta S = \sigma SZ \sqrt{\delta t} + O(\delta t)) \]
\[ + \frac{\partial^2}{\partial S \partial t} (S, t) \delta t + \cdots \]

therefore

\[ \nu \approx \frac{\partial^2 V}{\partial S^2} (S, t) \delta S \]
\[ \approx \frac{\partial^2 V}{\partial S^2} \sigma SZ \sqrt{\delta t} \]
\[ \uparrow \]
Gamma $\Gamma$! — Transaction cost is related to $\Gamma$

and the expected transaction cost in one revision is

\[ \mathbb{E} [\kappa \mid \nu \mid S] = \sqrt{\frac{2}{\pi}} \kappa \sigma S^2 \left| \frac{\partial^2 V}{\partial S^2} \right| \sqrt{\delta t} \]
\[ \left( \because \mathbb{E} [\mid Z \mid] = \sqrt{\frac{2}{\pi}} \right) \]

$\implies$
The expected change in portfolio value:

\[
\mathbb{E} [\delta \Pi] = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \kappa \sigma S^2 \sqrt{\frac{2}{\pi \delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right| \right) \delta t
\]

If the expected return is the same as the money in the bank, i.e.,

\[r \Pi \delta t = r (V - \Delta S) \delta t\]

Therefore,

\[r (V - \Delta S) = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \kappa \sigma S^2 \sqrt{\frac{2}{\pi \delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right| \right)\]

i.e.,

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \kappa \sigma S^2 \sqrt{\frac{2}{\pi \delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right| + r S \frac{\partial V}{\partial S} - r V = 0. \tag{5}
\]

Note that

1. Eq. (5) is a nonlinear PDE, i.e., in the presence of transaction costs, the value of a portfolio that is the sum of individual options is not the same as the sum of the values of individual options — Financial consequences:
   e.g., 2 call options with the same strike \(K\) and maturity on the same stock:
   
   One held long, the other held short
   \[\implies\text{ Net position zero.}\]
   
   (a) If hedged separately, lose money on both
   \[\implies\text{ Net loss at maturity due to transaction costs}\]
   
   (b) But, since they are the opposite positions, their hedgings are opposite, no need to hedge at all
   \[\implies\text{ No transaction costs}\]
   \[\implies\text{ Net balance }= 0 \text{ at maturity.}\]

2. If

\[
\frac{\partial^2 V}{\partial S^2} > 0
\]

(which is true, e.g., for a single call held long) then

\[
\left| \frac{\partial^2 V}{\partial S^2} \right| = \frac{\partial^2 V}{\partial S^2}
\]
thus
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \kappa \sigma S^2 \sqrt{\frac{2}{\pi \delta t} \frac{\partial^2 V}{\partial S^2}} + r S \frac{\partial V}{\partial S} - r V = 0.
\]

which gives rise to an effective volatility:
\[
\tilde{\sigma}^2 = \sigma^2 - 2 \kappa \sigma \sqrt{\frac{2}{\pi \delta t}}
\]

and we regain the Black-Scholes PDE with this effective volatility:
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \tilde{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0.
\]

3. If the call is held short, then
\[
\tilde{\sigma}^2 = \sigma^2 + 2 \kappa \sigma \sqrt{\frac{2}{\pi \delta t}}
\]
this is because a short position change all signs of the terms in our argument except the term associated with the transaction cost.

4. For small \(\kappa\), the difference of the portfolio values between zero transaction cost and no-zero transaction cost is
\[
\left. \frac{V(S,t)}{\partial \sigma} \right|_{\text{w/o TC}} - \left. \frac{V(S,t)}{\partial \sigma} \right|_{\text{with TC}} = \frac{\partial V}{\partial \sigma} (\sigma - \tilde{\sigma}) + \cdots
\]
\[
\approx \frac{\partial V}{\partial \sigma} \left( \kappa \sigma \sqrt{\frac{2}{\pi \delta t}} \right)
\]
\[
\uparrow V e g a \quad \text{— a correction term} \propto V e g a
\]

Note that in the computation above, we need the following
\[
\sigma^2 - \tilde{\sigma}^2 = 2 \kappa \sigma \sqrt{\frac{2}{\pi \delta t}} \quad \text{for a long call}
\]
\[
\therefore (\sigma - \tilde{\sigma}) (\sigma + \tilde{\sigma}) = 2 \kappa \sigma \sqrt{\frac{2}{\pi \delta t}}
\]
\[
\sigma \approx \tilde{\sigma}
\]
\[
\Rightarrow \sigma - \tilde{\sigma} = \kappa \sigma \sqrt{\frac{2}{\pi \delta t}}
\]
Since
\[
V e g a = \frac{S \sqrt{T - t}}{\sqrt{2 \pi}} \exp \left[ - \frac{d_1^2}{2} \right]
\]
\[
V (S, t) - \tilde{V} (S, t) \approx \frac{\kappa S}{\pi} \sqrt{\frac{T - t}{\delta t}} \exp \left[ -\frac{d_1^2}{2} \right]
\]
\[
= \frac{\kappa S}{\pi} \sqrt{\text{number of revision in} \ (T - t) \exp \left[ -\frac{d_1^2}{2} \right]}
\]

5. In general, \( \frac{\partial^2 V}{\partial S^2} \) is not single-signed, therefore

\[
\left| \frac{\partial^2 V}{\partial S^2} \right| \neq \frac{\partial^2 V}{\partial S^2}
\]

e.g., cf. bull-spread:

which, then requires numerical solutions.

2 The BS Equation and Heat Equation

Heat equation is

\[ u_t = u_{xx} \]

Changing variables

\[(S, t) \rightarrow (x, \tau)\]

with

\[
S = e^x, \quad \tau = \frac{1}{2} \sigma^2 (T - t)\]

\[
v(x, \tau) = V (S, t)
\]
Thus, we have
\[ x = \log S \]
and
\[ \frac{\partial x}{\partial S} = \frac{1}{S} = e^{-x} \]

Starting with the BS PDE:
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

\[
\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau} \frac{\partial \tau}{\partial t} = \frac{\partial V}{\partial \tau} \left( -\frac{1}{2} \sigma^2 \right)
\]
\[
\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial S} = \frac{\partial V}{\partial x} e^{-x}
\]
\[
\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial x} \left( \frac{\partial V}{\partial S} \right) \frac{\partial x}{\partial S} = \frac{\partial}{\partial x} \left( \frac{\partial V}{\partial x} e^{-x} \right) e^{-x}
\]
\[
= \frac{\partial^2 V}{\partial x^2} e^{-2x} - \frac{\partial V}{\partial x} e^{-2x}
\]

therefore,
\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = \frac{1}{2} \sigma^2 e^{2x} \left( \frac{\partial^2 V}{\partial x^2} e^{-2x} - \frac{\partial V}{\partial x} e^{-2x} \right)
\]
\[
= \frac{1}{2} \sigma^2 \left( \frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x} \right)
\]
\[
S \frac{\partial V}{\partial S} = e^{\alpha x} \frac{\partial V}{\partial x} e^{-x} = \frac{\partial V}{\partial x}
\]

\[\Rightarrow\]
\[
\frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} + \left( 1 - \frac{r}{2} \sigma^2 \right) \frac{\partial v}{\partial x} + \frac{r}{2} \sigma^2 v = 0
\]

which is an equation with constant coefficients.

Define:
\[ k \equiv \frac{r}{2} \sigma^2 \]
\[ v = e^{\alpha x + \beta \tau} u(x, \tau) \]

with \( \alpha, \beta \) to be determined. Then we have
\[ (\beta u + u_\tau) - (\alpha^2 u + 2\alpha u_x + u_{xx}) + (1 - k)(\alpha u + u_x) + ku = 0 \]

To eliminate \( u_x, u \)-terms, we set
\[ u_x : -2\alpha + (1 - k) = 0 \]
\[ u : \beta - \alpha^2 + (1 - k) \alpha + k = 0 \]
\[ \alpha = \frac{1 - k}{2}, \quad \beta = -\frac{(1 + k)^2}{4} \]

Hence,
\[ u_\tau = u_{xx} \]

which is the diffusion equation.

Note that

1. (a) \( \tau = \frac{1}{2} \sigma^2 (T - t), \quad \tau \in [0, +\infty) \), i.e., the final time \( t = T \) corresponds to the initial time for \( \tau = 0 \)

   (b) \( x = \log S \), therefore \( x \in (-\infty, +\infty) \)

2. The issue of numerical methods for \( u_\tau = u_{xx} \) vs. for the original BS PDE.

3. Use the fundamental solution, we can again obtain the option price.

   e.g., for a European call:
   \[ v (x, 0) = (e^x - K)_+ \]
   \[ u (x, 0) = e^{-\alpha x} v (x, 0) \]
   \[ = \left( e^{-\frac{1 + k}{2} x} (e^x - K)_+ \right) \]
   \[ = \left( e^{\frac{k}{2} (k+1)x} - K e^{\frac{k}{2} (k-1)x} \right)_+ \]
   \[ \equiv u_0 (x) \] (6)

   the solution of the heat equation:
   \[ u (x, \tau) = \int_{-\infty}^{\infty} u_0 (y) \left( \frac{1}{2 \sqrt{\pi \tau}} e^{-\frac{(x-y)^2}{4 \tau}} \right) dy \]

   Evaluating this integral, with Eq. (6), then transforming back to the variables \((S, t)\), yields
   \[ C (S, t) = SN (d_1) - Ke^{-r(T-t)} N (d_2) \]

4. Time-dependent parameters: The Black-Scholes PDE is valid as long as \( r, D, \sigma \) are known (deterministic) functions of time, (Note that, \( D (t) \) is related to the amount of dividend, \( D (t) S dt \) in a time step \( dt \)). The BS PDE is then
   \[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 (t) S^2 \frac{\partial^2 V}{\partial S^2} + (r (t) - D (t)) S \frac{\partial V}{\partial S} - r (t) V = 0 \] (7)

Let
\[ \bar{S} = S e^{\alpha(t)}, \quad \bar{V} = V e^{\beta(t)}, \quad \bar{t} = \gamma (t) \]
where $\alpha(t), \beta(t), \gamma(t)$ are chosen to eliminate time dependence, note that

\[
\frac{\partial V}{\partial t} = \frac{\partial}{\partial t} (Ve^{-\beta(t)}) = \left( \frac{\partial}{\partial t} V \right) e^{-\beta(t)} + V \frac{\partial}{\partial t} e^{-\beta(t)}
\]

\[
= \left[ \frac{\partial}{\partial t} \bar{V}(\bar{S}, \bar{t}) \right] + \frac{\partial}{\partial \bar{S}} \bar{V}(\bar{S}, \bar{t}) \left[ \frac{\partial \bar{S} e^{\alpha(t)}}{\partial \bar{t}} \right] e^{-\beta(t)} + \left( -\dot{\beta}(t) \right) V e^{-\beta(t)}
\]

\[
= \left[ \frac{\partial}{\partial \bar{t}} \bar{V}(\bar{S}, \bar{t}) \right] + \frac{\partial}{\partial \bar{S}} \bar{V}(\bar{S}, \bar{t}) \left[ \bar{S} \dot{\alpha}(t) \right] e^{-\beta(t)} + \left( -\dot{\beta}(t) \right) V e^{-\beta(t)}
\]

then the BS PDE (7) becomes

\[
\dot{\gamma}(t) \frac{\partial \bar{V}}{\partial \bar{t}} + \frac{1}{2} \sigma^2(t) \bar{S}^2 \frac{\partial^2 \bar{V}}{\partial \bar{S}^2} + \left( r(t) - D(t) + \dot{\alpha}(t) \right) \bar{S} \frac{\partial \bar{V}}{\partial \bar{S}} - \left( r(t) + \dot{\beta}(t) \right) \bar{V} = 0 \tag{8}
\]

therefore,

\[
\beta(t) = \int_t^T r(\tau) d\tau
\]

\[
\alpha(t) = \int_t^T \left[ r(\tau) - D(\tau) \right] d\tau
\]

\[
\gamma(t) = \int_t^T \sigma^2(\tau) d\tau
\]

then, Eq. (8) becomes

\[
\frac{\partial \bar{V}}{\partial \bar{t}} = \frac{1}{2} \bar{S}^2 \frac{\partial^2 \bar{V}}{\partial \bar{S}^2}
\]

which has coefficients that are time-independent. Suppose its solution is $\bar{V}(\bar{S}, \bar{t})$, then

\[
V = e^{-\beta(t)} \bar{V} \left( Se^{\alpha(t)}, \gamma(t) \right) \tag{9}
\]

is the solution of the time-dependent BS PDE (7).

5. Effective BS equation for a BS PDE with time-dependent coefficients:

If $V_{BS}$ is any solution of the BS PDE with constant $r_c, D_c$ and $\sigma_c$, then the solution satisfies

\[
V_{BS} = e^{-r_c(T-t)} \bar{V} \left( Se^{(r_c-D_c)(T-t)}, \sigma_c^2(T-t) \right) \tag{10}
\]

Comparing Solutions (9) and (10) gives the following results:
If
\[ r_c = \frac{1}{T - t} \int_t^T r(\tau) \, d\tau \]
\[ D_c = \frac{1}{T - t} \int_t^T D(\tau) \, d\tau \]
\[ \sigma^2_c = \frac{1}{T - t} \int_t^T \sigma^2(\tau) \, d\tau \]

are chosen, then

\[ V_{BS} \text{ and } V \text{ have the same value} \]

i.e., if time-averaged interest rate, dividend yield and volatility are chosen as above, European options with time-dependent parameters can be viewed as the the solution of an effective BS equation with time-independent parameters.

### 3 Barrier Options

A barrier option is like a European option except that it acquires or loses the whole value if the stock prices goes above (or below) a specified barrier \( X \).

1. Up-and-in option
   An **up-and-in** option pays off only if the stock price crosses \( X \) from below prior to maturity.

2. Down-and-in
   A **down-and-in** option pays off only if the stock price crosses \( X \) from above prior to maturity.
3. Up-and-out

An **up-and-out** option loses its value if the stock price crosses $X$ from below prior to maturity.

4. Down-and-out

A **down-and-out** option loses its value if the stock price crosses $X$ from above prior to maturity.

These are all **path-dependent** options.

Note that, e.g., a down-and-out **plus** a down-and-in call is a standard European call.
**Example:** pricing a down-and-out call.

Assume

\[ X < K \]
\[ X < S_0 \]

i.e., \( V(S,t) \) solves the BS PDE in the domain \( S > X \) with

the boundary conditions : \( V(X,t) = 0 \)
and the final condition : \( V(S,T) = (S - K)_+ \) at time \( t = T \)

It turns out that the solution can be explicitly expressed as

\[ V(S,t) = C(S,t) - \left( \frac{S}{X} \right)^{(1-k)} C\left( \frac{X^2}{S},t \right) \]

with

\[ k = \frac{r}{\frac{1}{2}\sigma^2} = \frac{2r}{\sigma^2} \]

where \( C(S,t) \) is the value of an ordinary European call with strike \( K \) and maturity \( T \).

Recall

\[ u_\tau = u_{xx} \]
\[ S > X \implies x > \log X \]
\[ V(S,t) = e^{\alpha x + \beta \tau} u(x, \tau) \]

\[ V = 0 \implies u = 0 \]
\[ V = (S - K)_+ \implies u_0(x) = e^{-\alpha x} (e^x - K)_+ \]
If we set the initial condition

\[ f_0(x) = -u_0(2 \log X - x) + u_0(x) \quad \forall x \]

i.e., which is odd-symmetric with respect to

\[ x = \log X \]

by reflection (Note that \( u_0(x) = 0 \) for \( x < \log X \),

then, the solution \( f(x, \tau) \) will have the odd symmetry about \( x = \log X \) also, i.e,

\[ f(x, \tau) = -f(2 \log X - x, \tau). \]

Note that \( f(x, \tau) \) has the initial data \( f(x, 0) = f_0(x) \) and \(-f(2 \log X - x, \tau)\) has the initial data:

\[
\begin{align*}
-f(2 \log X - x, 0) &= -f_0(2 \log X - x, 0) \\
&= -[-u_0(2 \log X - (2 \log X - x)) + u_0(2 \log X - x)] \\
&= +u_0(x) - u_0(2 \log X - x) \\
&= f_0(x)
\end{align*}
\]

therefore, \( f(x, \tau) \) will maintain

\[ f = 0 \quad \text{at } x = \log X \]
Since our BS PDE is linear, there exits a superposition principle:

solution 1: \( u_1(0) \rightarrow u_1(x,t) \)

solution 2: \( u_2(0) \rightarrow u_2(x,t) \)

\[ \implies u_1(0) + u_2(0) \rightarrow u_1(x,t) + u_2(x,t) \] is also a solution.

Note that

1. the initial data \( u_0(x) \) gives

\[ u_0(x) \rightarrow f_1(x,\tau) \]

then \( f_1(x,\tau) \) is just the same as the European call:

\[ f_1(x,\tau) = C(e^x, t) e^{-\alpha x - \beta \tau} \]

2. the initial data \( u_0(2 \log X - x) \) gives the solution

\[ f_2(x,\tau) = f_1(2 \log X - x) \]

therefore,

\[
e^{\alpha x + \beta \tau} f_2(x, \tau) = e^{\alpha x + \beta \tau} f_1(2 \log X - x) = e^{\alpha x + \beta \tau} e^{-\alpha(2 \log X - x) - \beta \tau} C(e^{2 \log X - x}, t)\left(\text{N.B. } S = e^x, \alpha = \frac{1-k}{2}, 2\alpha = 1-k\right)
\]

\[ = X^{-2\alpha} S^{2\alpha} C\left(\frac{X^2}{S}, t\right) \]

\[ = \left(\frac{S}{X}\right)^{1-k} C\left(\frac{X^2}{S}, t\right) \]

Hence,

\[ V(S, t) = C(S, t) - \left(\frac{S}{X}\right)^{1-k} C\left(\frac{X^2}{S}, t\right) \]