1 Parameterization of Binomial Models and Derivation of the Black-Scholes PDE.

Previously we treated binomial models as a pure theoretical toy model for our complete economy. We turn to the issue of how to estimate parameters in the model and demonstrate the power of the binomial model as a pricing tool — we will derive the famous Black-Scholes equation in this binomial setting.

1.1 The Market:
Consider our stock:

- Time at $t$, its value is $S_0$
- Time at $t+\delta$, its pdf is $p(S)$

Here, we consider a very short time interval $\delta$, i.e., $\delta \ll 1$.

\[
\text{Therefore, at } t = \delta, \text{ the expected value of the stock price is } \\
\mathbb{E}(S) = \int Sp(S) dS
\]

and its variance is

\[
\text{Var}(S) = \mathbb{E}[(S - \mathbb{E}(S))^2] \\
= \int (S - \mathbb{E}(S))^2 p(S) dS
\]

Next, we prescribe a model for our stock price movement:

1. Assumption 1:

\[
\mathbb{E}\left(\frac{S}{S_0}\right) = e^{\mu \delta}
\]

i.e., the rate of the expected return is $\mu$. 

2. Assumption 2:

\[ \text{Var} \left( \frac{S}{S_0} \right) = \sigma^2 \delta \]

where \( \sigma \) is the volatility.

N.B. Assumption 2 is a consequence of the following theorem, roughly stated, that if the total volatility over time is bounded below away from zero and bounded above, and if there is always some (finite) volatility over any time interval, then

\[ \text{Var} (S) \sim \delta \]

(see Neftci’s book, pp164-167).

Consider the binomial model

\[
\begin{array}{c}
\text{\textbullet} \quad S_0 \\
\text{\textbullet} \quad p \quad \quad u \quad S_0 \quad u \\
\text{\textbullet} \quad (1-p) \quad d \quad S_0 \quad d \\
\end{array}
\]

where \( p \) is a subjective probability.

1.2 Parameterization:

We want to match the mean and variance of this binomial model with those of our market, i.e.,

mean: \( pS_0u + (1-p)S_0d = S_0e^{\mu\delta} \)

var: \( pS_0^2u^2 + (1-p)S_0^2d^2 - (pS_0u + (1-p)S_0d)^2 = S_0^2\sigma^2\delta \)

N.B. \( \text{Var} (X) = \mathbb{E} (X^2) - [\mathbb{E} (X)]^2 \). Therefore,

\[
\begin{align*}
pu + (1-p)d &= e^{\mu\delta} \quad (1) \\
pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 &= \sigma^2\delta \quad (2)
\end{align*}
\]

thus, we have 2 equations, 3 unknowns: \( p, u, d \). There is a freedom of imposing constraints in our parameterization. Eq. (1) gives

\[
p = \frac{e^{\mu d} - d}{u - d}
\]

Substitute this \( p \) into Eq. (2) yields

\[
e^{\mu \delta} (u + d) - ud - e^{2\mu \delta} = \sigma^2\delta \quad (3)
\]

Next we are going to impose some specific constraints to obtain expressions for \( u, d, p \).
1.2.1 The First Choice: 

Impose \( ud = 1 \).

To the accuracy of \( O(\delta) \), we have

\[
\begin{align*}
u &= e^{+\sigma \sqrt{\delta}} \\
d &= e^{-\sigma \sqrt{\delta}}
\end{align*}
\]

and the corresponding risk-neutral probability is

\[
q = \frac{e^{r \delta} - d}{u - d} = \frac{e^{r \delta} - e^{-\sigma \sqrt{\delta}}}{e^{+\sigma \sqrt{\delta}} - e^{-\sigma \sqrt{\delta}}}
\]

which is independent of \( \mu \).

The no-arbitrage principle dictates the following risk-neutral valuation of a contingent claim with payoff \( f(S_T) \):

The present value \( f_0 = e^{-r \delta} [qf(S_0u, t + \delta) + (1 - q) f(S_0d, t + \delta)] \)

in which the rate of the expected market return \( \mu \) does not appear.

**A Deep Look at This Price:** Since

\[
\begin{align*}
f(S_0u, t + \delta) &= f(S_0 + S_0u - S, t + \delta) \\
&= f(S_0 + S_0(u - 1), t + \delta) \\
&= f\left(S_0 + S_0 \left(e^{+\sigma \sqrt{\delta}} - 1\right), t + \delta\right),
\end{align*}
\]

\[
f_0 = e^{-r \delta} \left\{ qf\left(S_0 + S_0 \left(e^{+\sigma \sqrt{\delta}} - 1\right)\right) + (1 - q) f\left(S_0 + S_0 \left(e^{-\sigma \sqrt{\delta}} - 1\right)\right) \right\}
\]

in which, for simplicity of notation, we have dropped \( t + \delta \). Taylor-expansion to \( O(\delta) \):

\[
f_0 = e^{-r \delta} \left\{ q \left[ f(S_0) + \frac{\partial f}{\partial S_0} S_0 \left(e^{+\sigma \sqrt{\delta}} - 1\right) + \frac{1}{2} \frac{\partial^2 f}{\partial S_0^2} S_0^2 \left(e^{+\sigma \sqrt{\delta}} - 1\right)^2 \right] \\
+ (1 - q) \left[ f(S_0) + \frac{\partial f}{\partial S_0} S_0 \left(e^{-\sigma \sqrt{\delta}} - 1\right) + \frac{1}{2} \frac{\partial^2 f}{\partial S_0^2} S_0^2 \left(e^{-\sigma \sqrt{\delta}} - 1\right)^2 \right] \right\}
\]

in which all the derivatives are evaluated at time \( t + \delta \). Note that
1. Since \( e^{-r\delta} [qS_0u + (1 - q) S_0d] = S_0 \), we have 
\[ qe^{\sigma\sqrt{\delta}} + (1 - q) e^{-\sigma\sqrt{\delta}} = e^{r\delta} \]

2. 
\[ (e^{\sigma\sqrt{\delta}} - 1)^2 \approx \sigma^2 \delta \]
\[ (e^{-\sigma\sqrt{\delta}} - 1)^2 \approx \sigma^2 \delta \]

Therefore, accurate to \( O(\delta) \), we obtain
\[ f_0 = e^{-r\delta} \left[ f(S_0, t + \delta) + \frac{\partial f}{\partial S_0} S_0 e^{r\delta} - \frac{\partial f}{\partial S_0} S_0 + \frac{1}{2} \sigma^2 \delta S_0^2 \right] \]

Using
\[ e^{-r\delta} \approx 1 - r\delta + O(\delta^2) \]
\[ e^{r\delta} \approx 1 + r\delta + O(\delta^2) \]

leads to
\[ f_0 = f(S_0, t + \delta) + \delta \left[ -rf(S_0, t + \delta) + rS_0 \frac{\partial f}{\partial S_0} + \frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 f}{\partial S_0^2} \right] \] (4)

accurate upto \( O(\delta) \). This formula gives the relation between the values of the contingent claim at time \( t \) and time \( t + \delta \).

1.2.2 Black-Scholes Equation:

Finally, we note that Eq. (4) can be rewritten as
\[ \frac{f(S_0, t + \delta) - f_0}{\delta} - rf(S_0, t + \delta) + rS_0 \frac{\partial f}{\partial S_0} + \frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 f}{\partial S_0^2} = 0 \]

As \( \delta \to 0 \), we arrive at the famous Black-Scholes PDE:
\[ \frac{\partial f}{\partial t} - rf + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = 0 \]

where the subscript 0 is dropped.
1.2.3 The Second Choice:

\[ p = \frac{1}{2} \]

Now Eqs. (1) and (2) become

\[
\begin{align*}
    u + d &= 2e^{\mu \delta} \\
    u^2 + d^2 &= 2\sigma^2 \delta + 2e^{2\mu \delta}
\end{align*}
\]

To the order of \( O(\delta) \), the solutions are

\[
\begin{align*}
    u &= e^{(\mu - \frac{1}{2}\sigma^2)\delta + \sigma \sqrt{\delta}} \\
    d &= e^{(\mu - \frac{1}{2}\sigma^2)\delta - \sigma \sqrt{\delta}}
\end{align*}
\]

For this binomial model, the associated risk-neutral probability is

\[
q = \frac{e^{r \delta} - d}{u - d} = \frac{e^{(r - \mu + \frac{1}{2}\sigma^2)\delta} - e^{-\sigma \sqrt{\delta}}}{e^{r \delta} - e^{-\sigma \sqrt{\delta}}}
\]

It appears that all parameters, \( u, d, q \) depend on \( \mu \). Does the risk-neutral pricing also depend on \( \mu \)? Let us examine this question. Consider a contingent claim with payoff \( f(S_T) \), according to the no-arbitrage valuation, its present value is

\[
f_0 = e^{-r \delta} \left[ qf(S_0u, t + \delta) + (1 - q)f(S_0d, t + \delta) \right] = e^{-r \delta} \left\{ qf(S_0 + S_0 \left( e^{(\mu - \frac{1}{2}\sigma^2)\delta + \sigma \sqrt{\delta}} - 1 \right) ) + (1 - q)f(S_0 + S_0 \left( e^{(\mu - \frac{1}{2}\sigma^2)\delta - \sigma \sqrt{\delta}} - 1 \right) ) \right\}.
\]

Again, for simplicity of notation, \( t + \delta \) is dropped from the second line above. Taylor expansion yields, up to \( O(\delta) \),

\[
f_0 = e^{-r \delta} \left\{ q \left[ f(S_0, t + \delta) + \frac{\partial f}{\partial S_0} S_0 \left( e^{(\mu - \frac{1}{2}\sigma^2)\delta + \sigma \sqrt{\delta}} - 1 \right) + \frac{1}{2} S_0^2 \frac{\partial^2 f}{\partial S_0^2} \left( e^{(\mu - \frac{1}{2}\sigma^2)\delta + \sigma \sqrt{\delta}} - 1 \right)^2 \right] \\
+ (1 - q) \left[ f(S_0, t + \delta) + \frac{\partial f}{\partial S_0} S_0 \left( e^{(\mu - \frac{1}{2}\sigma^2)\delta - \sigma \sqrt{\delta}} - 1 \right) + \frac{1}{2} S_0^2 \frac{\partial^2 f}{\partial S_0^2} \left( e^{(\mu - \frac{1}{2}\sigma^2)\delta - \sigma \sqrt{\delta}} - 1 \right)^2 \right] \right\} \tag{5}
\]

in which all the derivatives are evaluated at time \( t + \delta \). Note that

1. Again because \( e^{-r \delta} [qS_0u + (1 - q) S_0d] = S_0 \), we have

\[
qe^{(\mu - \frac{1}{2}\sigma^2)\delta + \sigma \sqrt{\delta}} + (1 - q)e^{(\mu - \frac{1}{2}\sigma^2)\delta - \sigma \sqrt{\delta}} = e^{r \delta}
\]
2.

\[
\left(e^{\left(\mu - \frac{1}{2}\sigma^2\right)\delta \pm \sigma \sqrt{\delta} - 1}\right)^2
\approx \left[1 + \left(\mu - \frac{1}{2}\sigma^2\right)\delta \pm \sigma \sqrt{\delta} + \frac{1}{2}\sigma^2\delta - 1\right]^2
= \left(\mu\delta \pm \sigma \sqrt{\delta}\right)^2
= \sigma^2\delta + O\left(\delta^{3/2}\right)
\]

Using the results in points 1-2 above, accurate to \(O(\delta)\), we obtain

\[
f_0 = (1 - r \delta) \left[ f(S_0, t + \delta) + \frac{\partial f}{\partial S_0} S_0 r \delta + \frac{1}{2} \sigma^2 \delta S_0^2 \frac{\partial^2 f}{\partial S_0^2} \right]
= f(S_0, t + \delta) + \delta \left[-rf(S_0, t + \delta) + rS_0 \frac{\partial f}{\partial S_0} + \frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 f}{\partial S_0^2}\right]
\tag{6}
\]

Note that

1. Due to the order shown in point 2 above, we only need to evaluate \(q\) up to \(O(1)\) to ensure the entire expression (5) is valid up to the order \(O(\delta)\). Therefore,

\[
q = \frac{e^{(r - \mu + \frac{1}{2}\sigma^2)\delta} - e^{\sigma \sqrt{\delta}}}{e^{\sigma \sqrt{\delta}} - e^{-\sigma \sqrt{\delta}}}
= \frac{1}{2} \left[1 - \frac{(\mu - r)}{\sigma} \sqrt{\delta}\right] + O(\delta)
= \frac{1}{2} + O\left(\delta^{1/2}\right)
\]

2. Eq. (6) is exactly the same price as we obtained in the case of the first choice, i.e., \(ud = 1\). In particular, we note that, although the parameters, \(u,d,q\), now depend on \(\mu\), the rate of the expected market returned. The risk-neutral pricing is still independent of \(\mu\) — it disappears into higher order terms, which will have no bearings on our no-arbitrage price.

3. We can again obtain the same Black-Scholes PDE, which describe how the value of a contingent claim will evolve in time.

4. A real miracle: different parameterizations lead to the same Black-Scholes PDE under the no-arbitrage principle for a given market.

5. For different markets (described by different values of \(\mu\)), as long as the market volatility \(\sigma\) is the same, we will get the same price for the option. The rate of the expected market return is irrelevant. This is a far stronger result than the case where, for a given set of \(u,d\), no-arbitrage pricing is independent of \(p\) as in our theoretical binomial toy model of economy.
1.3 The Simplest Binomial Model

Note that

\[
q = \frac{e^{(r-\mu + \frac{1}{2} \sigma^2)\delta} - e^{-\sigma \sqrt{\delta}}}{e^{\sigma \sqrt{\delta}} - e^{-\sigma \sqrt{\delta}}}
\]
\[
= \frac{1}{2} \left[ 1 - \frac{(\mu - r)}{\sigma \sqrt{\delta}} \right] + O(\delta).
\]

If we choose

\[
\mu = r
\]

then

\[
q = \frac{1}{2} + O(\delta)
\]

The \( O(\delta) \)-term in \( q = \frac{1}{2} + O(\delta) \) will not affect our risk-neutral pricing or the derivation of the Black-Scholes PDE because its combined contribution with \( f(S_u, t + \delta) \) and \( f(S_d, t + \delta) \) to the price is higher order than \( O(\delta) \). Since \( p = 1/2 \), we can simply choose \( p = q = 1/2 \). For risk-neutral valuation, the simplest binomial model for a short duration \( \delta \) with a price accuracy within \( O(\delta) \) is

\[
\begin{align*}
S_0 & \leq e^{(r - \frac{1}{2} \sigma^2)\delta + \sigma \sqrt{\delta}} \\
S_0 & \geq e^{(r - \frac{1}{2} \sigma^2)\delta - \sigma \sqrt{\delta}}
\end{align*}
\]
2 Multiperiod Binomial Trees

2.1 Setup:

1. 2 securities:
   (a) a risky asset (e.g., stock without dividend);
   (b) a risk-free asset (e.g., bond).

2. A series of times:

\[ 0, \delta t, 2\delta t, \ldots, N\delta t = T \]

at which trades take place.

3. A binomial tree of possible states for stock prices

\[
\begin{array}{c}
S_j \\
\downarrow \\
\begin{array}{c}
p_j \quad S_{2j+1} \\
(1-p_j) \quad S_{2j}
\end{array}
\end{array}
\]

and constant interest rate \( r \) (can be easily generalized to \( r_k \) for time interval \( k\delta t \).

1. Since the market permits no-arbitrage, we have

\[ s_{2j^*} < e^{\delta t} s_j < s_{2j+1} \quad \forall j \]

This is an example of dynamically complete market.

2.2 Generalization:
Recombinant Tree:

Note that: At time step $n$,

1. There are $2^n$ states for the non-recombinant tree;
2. There are $(n + 1)$ states for the recombinant tree.

This fact gives rise to the numerical advantage of recombinant trees.

Note that the parameterization with $u, d$ gives a naturally recombined tree:

Risk-neutral valuation for each node:

\[
\begin{align*}
    f_{now} &= e^{-r\delta t} (q f_{up} + (1 - q) f_{down}) \\
    q &= \frac{e^{r\delta t} S_{now} - S_{down}}{S_{up} - S_{down}}
\end{align*}
\]

with the replicating portfolio:

- **Stock**: $\Delta$ shares, $\Delta = \frac{f_{up} - f_{down}}{S_{up} - S_{down}}$
- **Bond**: $f_{now} - \Delta S_{now}$

Note that $\phi \equiv \Delta$. 
2.2.1 Evaluation is just "working backward through the tree":

Example:

For simplicity, \( r = 0 \), and the price movement is assumed to be such that \( q = \frac{1}{2} \) at every node for this tree.

What is the value of a European call option with \( K = 100 \) at \( T = 3\delta t \)?

Note that the value at maturity is

\[
(S_T - 100)_+ = 60, 20, 0, 0
\]

At the node with \( S = 140 \):

\[
f = qf_{up} + (1 - q)f_{down} = \frac{1}{2} \times 60 + \frac{1}{2} \times 20 = 40
\]

Working backward, we conclude the value of the option is 15.

Why this is the correct price?
Because it can be replicated at every trading time.

The replication (in one possible path) — Consider that a bank sells the option:

1. At time \( t = 0 \),

\[
\Delta = \frac{25 - 5}{120 - 80} = 0.5 \quad \text{shares of stock}
\]

stock : \( \Delta \times S = 0.5 \times 100 = 50 \)

Bond : \( f_{\text{now}} - \Delta \times S = 15 - 50 = -35 \)
i.e., the bank sells the option for $15

borrows $35

uses these $15 + $35 = $50
to buy $\frac{1}{2}$ shares of the stock.

2. Next step: say, the stock goes up to 120. the the new $\Delta$ is

$$\Delta = \frac{40 - 10}{140 - 100} = 0.75$$

Need additional 0.25 shares of stock to be purchased at the present value of stock, $120.

$$\implies \text{Need } 120 \times 0.25 = 30 \text{ borrowed}$$

Debt = $35 + 30 = $65.$

3. If the stock, say, goes up to $140, then

$$\Delta = \frac{60 - 20}{160 - 120} = 1$$

Need another 0.25 shares at $140/$share:

$$0.25 \times 140 = 35$$

Debt: $35 + 65 = 100.$

4. If the stock, say, goes down to $120 (at maturity), then,

The debt = $100, which matches the strike $K$

Portfolio: $\underbrace{120}_{\text{1 Share of Stock}} - 100 = 20$, which replicates the option claim.

Thus, replicating the claim. That is, the bank sells the option at $t = 0$; At time $t = T$, it could deliver one share of stock, collect $K = 100$, pay off the loan. **No gain, no loss!**

**Conclusion:** The portfolio is *self-financing* at each trading, i.e., the total value of the portfolio before and after each trade are the same.

This can be again seen as follows:

For simplicity, assume interest rate $r = 0$. Notation: At tick-time $i$,

A portfolio $\Pi_i : \Delta_{i+1} \text{ stock}$

$$\psi_{i+1} = f_i - \Delta_{i+1} S_i$$

1. Start:

$$\Pi_0 : \Delta_1 S_0 + \psi_1 = f_0$$
2. One tick:

\[
\Pi_1 : \text{worth } \psi_1 + \Delta_1 S_1 \\
= \psi_1 + \Delta_1 S_0 - \Delta_1 S_0 + \Delta_1 S_1 \\
= f_0 + \Delta_1 (S_1 - S_0) \\
= f_0 + \frac{f_1 - f_0}{S_1 - S_0} (S_1 - S_0) \\
= f_1
\]

rebalance \( \Delta \) such that

\[ f_1 = \Delta_2 S_1 + \psi_2 \]

3. Tick-time 2:

\[
\Pi_2 : \text{worth } \Delta_2 S_2 + \psi_2 \\
= f_1 + \Delta_2 (S_2 - S_1) \\
= f_1 + \frac{f_2 - f_1}{S_2 - S_1} (S_2 - S_1) \\
= f_2
\]

4. At \( t = T \):

\[
\Pi_T : \Delta_{T-1} S_T + \psi_{T-1} \\
= f_{T-1} + \Delta_{T-1} (S_T - S_{T-1}) \\
= f_T
\]

which produces the claim.

Thus, the replicating process is self-financing.
2.3 Valuation Formula:

2.3.1 Case of General Trees:

\[ q_1 = \frac{e^{r \delta t} S_1 - S_2}{S_3 - S_2} \]
\[ q_2 = \frac{e^{r \delta t} S_2 - S_4}{S_5 - S_4} \]
\[ q_3 = \frac{e^{r \delta t} S_3 - S_6}{S_7 - S_6} \]

Therefore,

\[ f(3) = e^{-r \delta t} [q_3 f(7) + (1 - q_3) f(6)] \]
\[ f(2) = e^{-r \delta t} [q_2 f(5) + (1 - q_2) f(4)] \]

\[ \Rightarrow \]
\[ f(1) = e^{-r \delta t} [q_1 f(3) + (1 - q_1) f(2)] \]
\[ = e^{-2r \delta t} [q_1 q_3 f(7) + q_1 (1 - q_3) f(6) + (1 - q_1) q_2 f(5) + (1 - q_1)(1 - q_2) f(4)] \]

i.e.,

Initial value of a claim
\[ = e^{-rN \delta t} \sum_{\text{final states}} (\text{the probability of the path associated with a final state } i) \times (\text{payoff of state } i) \]
2.3.2 Case of Recombinant Trees

\[ q = \frac{e^{r \delta t} - d}{u - d} \]

\[ f(1) = e^{-2r \delta t} \left[ q^2 f(7) + 2q(1-q)f(5) + (1-q)^2 f(4) \right] \]

For the \( N \)-step:

The present value of an option with payoff \( f(S_T) \):

\[ e^{-rN \delta t} \sum_{k=0}^{N} \left( \binom{N}{k} q^k (1-q)^{N-k} f(S_0 u^k d^{N-k}) \right) \]

where \( \binom{N}{k} \) is the number of ways of having \( k \) steps up and \( N - k \) steps down in a total \( N \) time-steps.

E.g., A European call has the present value:

\[ e^{-rN \delta t} \sum_{k=0}^{N} \left( \binom{N}{k} q^k (1-q)^{N-k} (S_0 u^k d^{N-k} - K)_+ \right) \]