1 Binomial, Trinomial and More General One-Period Models

Explore the consequences of no arbitrage principle

1.1 The Binomial Model

Consider the economy:

1. Only 2 securities:
   (a) a stock (no-dividend paying): price \( s_1 \) at present
   (b) a bond (interest rate \( r \)) \( e^{-rT} \) at present and $1 at date \( T \)

2. Only 1 maturity date \( T \), i.e., the so-called one-period;

3. Only 2 possible states for the stock price at \( T \):
   \( S_T = s_2 \) or \( s_3 \)
   \( s_3 > s_2 \) \quad s_2 : \text{lower state}; \quad s_3 : \text{up state}

N.B. We will show that the subjective probability \( p \) in the sense of a normal market does not enter valuation of options!

Consider a contingent claim whose payoff function is \( f(S_T) \).
For example, if $f$ is a long call, then

$$
\begin{align*}
f_2 &= (s_2 - K)_+ \quad \text{if the lower state is realized;}
f_3 &= (s_3 - K)_+ \quad \text{if the up state is realized.}
\end{align*}
$$

The old "wrong" thinking (However "reasonable"):

\[
\text{the value of } f = e^{-rT} [p f_3 + (1 - p) f_2] = e^{-rT} \mathbb{E}_p [f(S_T)]
\]

Let us reveal why this is wrong:
First, assumption: The economy permits no arbitrage. Therefore

$$s_2 < s_1 e^{rT} < s_3$$

Why?

1. (a) if $s_3 > s_2 \geq s_1 e^{rT}$,
   
   At $t = 0$, borrow $s_1$ to buy a stock
   
   At $T$, the stock worth $s_2$ or $s_3$. Since both are greater than $s_1 e^{rT}$, pay back $s_1 e^{rT}$ and left with a net profit:
   
   at least $s_2 - s_1 e^{rT}$ without any risk

   (b) Even if $s_2 = s_1 e^{rT}$, if $s_3$ occurs with a nonzero probability, then there is a profit without risk.

   (c) Similarly, if $s_1 e^{rT} \geq s_3 > s_2$, then there is an arbitrage opportunity.

Now let's turn to the question of valuation:
Consider the portfolio:

$\phi$ shares of stock;

$\psi$ bonds.

initial value: $\phi s_1 + \psi e^{-rT}$
Demanding the value at date $T$ such that
\[
\begin{align*}
\phi s_2 + \psi \cdot 1 &= f_2 \\
\phi s_3 + \psi \cdot 1 &= f_3
\end{align*}
\] (1)
i.e., replicating the contingent claim.

Eq. (1) has 2 unknowns $\phi, \psi$ and can be easily solved:

\[\phi = \frac{f_3 - f_2}{s_3 - s_2}, \quad \psi = \frac{s_3 f_2 - s_2 f_3}{s_3 - s_2}\]

Therefore, the initial value of the replicating portfolio is

\[
V(f) = \phi s_1 + \psi e^{-rT}
= \left(\frac{f_3 - f_2}{s_3 - s_2}\right) s_1 + \left(\frac{s_3 f_2 - s_2 f_3}{s_3 - s_2}\right) e^{-rT}
= e^{-rT} \left[ \left(\frac{s_1 e^{rT} - s_2}{s_3 - s_2}\right) f_3 + \left(\frac{s_3 - s_1 e^{rT}}{s_3 - s_2}\right) f_2 \right]
\]
i.e.,

\[
V(f) = e^{-rT} [q f_3 + (1 - q) f_2]
\]
\[q = \frac{s_1 e^{rT} - s_2}{s_3 - s_2}\]

Note that

1. Since the economy permits no arbitrage, i.e., $s_2 < s_1 e^{rT} < s_3$, then

\[0 < q < 1\]

2. Why is $V(f)$ a correct price for the contingent claim?

If one prices the contingent claim higher than $V(f)$, then you sell the contingent claim and use the portion of the profit to setup the replicating portfolio.

On date $T$, since the replicating portfolio produces the exact same amount of payoff as the contingent claim, you can use the replicating portfolio to cover the claim regardless of which state (up or lower) of the economy is in.

Hence, there is a net profit with certainty and without any risk! Therefore, in order to avoid arbitrage, the price cannot be higher than $V(f)$. A similar no-arbitrage argument leads to the conclusion that it cannot be lower than $V(f)$.

1) $V(f)$ is the no-arbitrage price of $f$.

2) the value of the replicating portfolio is the no-arbitrage price of the contingent claim.
3. Another way of verifying \( V(f) \) is the no-arbitrage price:

Because the no-arbitrage argument leads to

(a) a portfolio with non-negative payoff must have a non-negative value;
(b) a portfolio with non-negative and sometimes positive payoff must have a positive value.

Let us check:

\[
\begin{align*}
1) & \quad f_2, f_3 \geq 0 \\
& \quad \Rightarrow V(f) = qf_3 + (1 - q)f_2 \geq 0 \\
2) & \quad f_3 > 0, f_2 \geq 0 \\
& \quad \Rightarrow V(f) = qf_3 + (1 - q)f_2 > 0
\end{align*}
\]

Therefore, \( V(f) \) is the no-arbitrage price of \( f \). Statements (a) and (b) are necessary and sufficient for no-arbitrage — the Arbitrage Theorem below will make this point even more transparent.

4. \( V(f) \) is independent of \( p \), i.e., it does not depend on whether you’re a pessimist or an optimist.

\[
V(f) = e^{-rT}E_q(f(S_T))
\]

i.e.,

\[
\begin{align*}
f_1 & \equiv V(f) = e^{-rT}[qf(s_3) + (1 - q)f(s_2)] \\
& = e^{-rT}[qf_3 + (1 - q)f_2]
\end{align*}
\]

with

\[
q = \frac{s_1e^{rT} - s_2}{s_3 - s_2} \quad \text{— risk-neutral probability.}
\]

where \( f_1 \) denotes the present value of \( f \).

Note that \( q \) is determined by \( s_3, s_2, s_1, r \) only. It has nothing to do with the subjective probability \( p \)!

In general, \( f_1 = V(f) \) is not

\[
W(f) \equiv e^{-rT}(pf_3 + (1 - p)f_2) = e^{-rT}E_p[f(S_T)]
\]

e.g.,

\[
p \lesssim 1 \quad \text{then} \quad W(f) \approx e^{-rT}f_3
\]
1.1.1 Hedging and replication:

A first look at hedging in a 2 state world.

Consider the portfolio:

\[
\begin{align*}
\text{short} : & \quad -1 \quad \text{unit of the claim} \quad f \\
\text{long} : & \quad \Delta \quad \text{units of stock with price} \quad s_1
\end{align*}
\]

\[
\Delta s_1 - f_1 < \Delta s_3 - f_3
\]

\[
\Delta s_2 - f_2
\]

The uncertainty:

\[
(\Delta s_3 - f_3) - (\Delta s_2 - f_2)
\]

No uncertainty (i.e., risk-free or risk-neutral):

\[
(\Delta s_3 - f_3) - (\Delta s_2 - f_2) = 0
\]

therefore

\[
\Delta = \frac{f_3 - f_2}{s_3 - s_2} \quad \text{— Delta hedge}
\]

Note that

\[
\Delta \equiv \phi
\]

**If a portfolio has no risk, it can grow only at risk-free rate**, therefore, the value of the portfolio at \( t = 0 \) is related to the value of the portfolio at \( t = T \) by the discount factor \( e^{-rT} \) of a risk-free asset, i.e.,

\[
(\Delta s_1 - f_1) = e^{-rT}(\Delta s_3 - f_3).
\]

Solving for \( f_1 \) and plugging the expression of \( \Delta \), after algebraic re-arrangement, we get the present value of the contingent claim

\[
f_1 = e^{-rT}[q f_3 + (1 - q) f_2]
\]

with the same

\[
q = \frac{s_1 e^{rT} - s_2}{s_3 - s_2}
\]

Therefore, the risk-neutral pricing is the no-arbitrage pricing.

**Conclusions:**

1. No-arbitrage condition determines the value of \( f \).
2. Replicating portfolio gives the no-arbitrage price.
3. Risk-neutral valuation is the no-arbitrage pricing.

Comment: A market in which no-arbitrage determines the value of contingent claims is called a "complete" market.

1.1.2 Some important relations:

1. We have the algebraic identity:

\[ s_1 = e^{-rT} [qs_3 + (1 - q)s_2] \]

i.e., the stock price satisfies the same relation as \( f \).

Verify:

\[ e^{-rT} \left[ \left( \frac{s_1 e^{rT} - s_2}{s_3 - s_2} \right) s_3 + \left( 1 - \frac{s_1 e^{rT} - s_2}{s_3 - s_2} \right) s_2 \right] = s_1 \]

Therefore,

\[ e^{-rT} \mathbb{E}_q [S_T] = s_1 \]

or

\[ \mathbb{E} [e^{-rT} S_T] = e^{-r} s_1 \]

in the form of

\[ \mathbb{E} [X_T] = X_s, \quad s < T \]

which defines the so-called martingale process.

2. Miracle: Both the underlying asset \( S \) and the derivative \( f \) satisfy

\[ \mathbb{E}_q \left[ \frac{f_T}{f_1} \right] = \mathbb{E}_q \left[ \frac{S_T}{S_1} \right] = e^{rT} \]

where \( f_T \equiv f(S_T) \). That is, with respect to the risk-neutral probability \( q \), the expected return of both \( s \) and \( f \) has the risk-free rate. We will further reveal the deep meaning of these equations.

In general, with respect to the subjective probability of market movement,

\[ \mathbb{E}_p \left[ \frac{S_T}{S_1} \right] > e^{rT} \]

otherwise, why do you want to invest? Usually, we expect

(a)

\[ \mathbb{E}_p \left[ \frac{S_T}{S_1} \right] = e^{(r + r')T} \]

where \( r' \) is the risk premium — a reward for taking risk.

(b)

\[ \mathbb{E}_p \left[ e^{-rT} S_T \right] > e^{-r} s_1 = s_1 \]

i.e., \( X_T \equiv e^{-rT} S_T \) would be a submartingale.
1.2 Arbitrage Theorem

1.2.1 Reformulation (Baby version of Arbitrage Theorem)

Let

\[ q_3 \equiv q, \quad q_2 \equiv 1 - q \]

Let

\[ Q_3 \equiv q_3 e^{-rT}, \quad Q_2 \equiv q_2 e^{-rT} \]

\[ \therefore q_2 + q_3 = 1 \]
\[ \therefore Q_2 + Q_3 = e^{-rT} \]

therefore

\[ s_1 = e^{-rT} (q_3 s_3 + q_2 s_2) = Q_3 s_3 + Q_2 s_2 \]
\[ f_1 = e^{-rT} (q_3 f_3 + q_2 f_2) = Q_3 f_3 + Q_2 f_2 \]

Rewrite all these:

\[
\begin{pmatrix}
  e^{-rT} \\
  s_1 \\
  f_1
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 \\
  s_2 & s_3 \\
  f_2 & f_3
\end{pmatrix}
\begin{pmatrix}
  Q_2 \\
  Q_3
\end{pmatrix}
\]

(2)

Now regarding \( Q_2, Q_3 \) as unknowns in Eq. (2). The baby-version of the arbitrage theorem reads:

1. If there is no arbitrage opportunity, then there exists a positive solution \( Q_2 > 0, Q_3 > 0 \) for Eq. (2)

2. If Eq. (2) has positive solutions, \( Q_2 > 0, Q_3 > 0 \), then there is no arbitrage opportunity;

In this case, we know there is a solution \( q_2 > 0, q_3 > 0 \) (or \( Q_2 > 0, Q_3 > 0 \)). The theorem holds obviously.
1.2.2 Arbitrage Theorem in the General One-period Market Models

1. N securities, $i = 1, 2, \cdots, N$, the present value is represented by

$$ S = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{pmatrix} $$

2. K final states, $\alpha = 1, 2, \cdots, K$, the payoff matrix is

$$ D = \begin{pmatrix} d_{11} & \cdots & d_{1K} \\
\vdots & \ddots & \vdots \\
d_{N1} & \cdots & d_{NK} \end{pmatrix} $$

State 1 $\uparrow$ $\cdots$ $\uparrow$ State $K$

3. Portfolio is described by

$$ \theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix} $$

— the position in security 1

$\cdots$

— the position in security $N$

Therefore, the total investment:

$$ \sum_{i=1}^{N} s_i \theta_i = S^T \theta $$

and the payoff in state $\alpha$ is

$$ \sum_{i=1}^{N} d_{i\alpha} \theta_i. $$

which is an element in the matrix

$$ D^T \theta = \begin{pmatrix} d_{11} & \cdots & d_{N1} \\
\vdots & \ddots & \vdots \\
d_{1K} & \cdots & d_{NK} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix}. $$

$D^T \theta$ is a payoff matrix, listing the payoffs in all possible states.

**Definition of Arbitrage portfolio:** If *either* one of the following condition is satisfied:

1. $S^T \theta \leq 0$ and $D^T \theta > 0$, or
2. $S^T \theta < 0$ and $D^T \theta \geq 0$.

**No Arbitrage Principle:**
1. $D^T \theta \geq 0 \implies S^T \theta \geq 0$; and
2. $D^T \theta \geq 0$ and $S^T \theta = 0 \implies D^T \theta = 0$

**Arbitrage Theorem:**
1. If there is no arbitrage opportunity, then, there exists a solution $Q = \begin{pmatrix} Q_1 \\ \vdots \\ Q_K \end{pmatrix} > 0$

   for the equation $S = DQ$;
2. If $S = DQ$ holds for some $Q > 0$, then, there is no arbitrage opportunity.

Proof is a direct consequence of Farkas-Minkowski Theorem in Linear Algebra (see any good textbook for the FM theorem).

**Consequences of the theorem:**
1. If Security 1 is a bond, then,

   $$\begin{pmatrix} e^{-rT} \\ s_2 \\ \vdots \\ s_N \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_{21} & d_{22} & \cdots & d_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N1} & \cdots & d_{NK} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_K \end{pmatrix}$$

   then

   $$e^{-rT} = Q_1 + Q_2 + \cdots + Q_K$$

   $$q_i = e^{rT}Q_i$$

   $$\therefore 1 = q_1 + q_2 + \cdots + q_K$$

   which gives a probability-like quantity — the risk-neutral measure (probability). Now the other securities (i.e., elements in other rows) satisfy

   $$s_i = e^{-rT}(q_1d_{i1} + q_2d_{i2} + \cdots + q_Kd_{iK}) \quad \forall i$$

   That is,

   $$V_{\text{now}} = e^{-rT}\mathbb{E}_q[V_T]$$
where $V$ stands for values of stocks and their derivatives. Clearly the importance of bonds is reflected in the condition

$$q_1 + q_2 + \cdots + q_K = 1$$

or the existence of the risk-neutral measure.

2. Note that the theorem does not say anything about uniqueness of the solution $Q$. The consequence of this is that the no arbitrage principle can set a bound on the prices of derivatives rather than determine the prices of derivatives in an incomplete market (we will discuss this below).

Proof of the Arbitrage Theorem:
Part 2: If

$$S = DQ,$$

then

$$S^T \theta = Q^T D^T \theta$$

$$\therefore (1) \quad D^T \theta \geq 0 \implies S^T \theta \geq 0$$

— this is Statement 1) in the No-Arbitrage Principle;

$$\therefore (2) \quad D^T \theta \geq 0, S^T \theta = 0 \text{ yields } D^T \theta = 0$$

— this is Statement 2) in the No-Arbitrage Principle;

otherwise, $Q^T D^T \theta \neq 0$ (since all the elements in $Q$ is positive), i.e., there are some no-zero elements in $Q^T D^T \theta$. But all elements in $S^T \theta$ vanishes. This leads to contradiction.

Part 1: Farkas-Minkowski theorem says

$$Q > 0, \quad S = DQ \quad \text{iff} \quad D^T \theta \geq 0 \implies S^T \theta \geq 0$$

Statement 2) in the No-Arbitrage Principle is a consequence of the following:

$$\therefore \text{ statement } 1 \implies S = DQ, \ Q > 0$$

$$\therefore S^T \theta = Q^T D^T \theta$$

if $D^T \theta \geq 0$ and $S^T \theta = 0 \implies D^T \theta = 0$ i.e., statement 2) holds.

QED.

1.3 Incomplete Market/Nonreplicatibility

A special case of this condition — trinomial model:

1. one-period $T$;
2. two securities: a stock and a bond;

3. three final states: \( s_4 > s_3 > s_2 \)

Again, a no-arbitrage argument leads to
\[ s_2 < s_1 e^{rT}, \text{ and } s_1 e^{rT} < s_4 \]

for this economy.

Consider the contingent claim:

Is this contingent claim replicatable? i.e., can we find solutions of the following system of equations
\[
\begin{align*}
\phi s_2 + \psi &= f_2 \\
\phi s_3 + \psi &= f_3 \\
\phi s_4 + \psi &= f_4
\end{align*}
\]

for \((\phi, \psi)\)? In general, there is no solution. Therefore,

1. This market is not complete;

2. Most of contingent claims are not replicatable in this market.

But, the no-arbitrage argument still sets bounds for the prices of \( f \).

From the arbitrage theorem, we know
\[
\exists q_2, q_3, q_4 > 0 \text{ such that } \\
f_1 = e^{-rT} (q_4 f_4 + q_3 f_3 + q_2 f_2) \\
\text{with } s_1 = e^{-rT} (q_4 s_4 + q_3 s_3 + q_2 s_2) \\
\text{and } 1 = q_2 + q_3 + q_4
\]
Since \((q_2, q_3, q_4)\) is not unique, for the present value \(V(f) \equiv f_1\), we have

\[
\min_{s_1 = e^{-rT}(q_4s_4 + q_3s_3 + q_2s_2) \quad q_2 + q_3 + q_4 = 1 \quad q_2, q_3, q_4 > 0} \left\{ e^{-rT} (q_4f_4 + q_3f_3 + q_2f_2) \right\} \leq V(f) \leq \max_{s_1 = e^{-rT}(q_4s_4 + q_3s_3 + q_2s_2) \quad q_2 + q_3 + q_4 = 1 \quad q_2, q_3, q_4 > 0} \left\{ e^{-rT} (q_4f_4 + q_3f_3 + q_2f_2) \right\}
\]

Comment:

From the above examples of binomial and trinomial models, we see that whether or not one can uniquely price a derivative using the no-arbitrage principle depends on what market model one uses to describe the dynamics of the underlying assets — wherein lies the peril and opportunity of money-making.