1 Martingales and Their Significance in Option Pricing

1.1 Martingales

\[ S_{\text{now}} = e^{-r\delta t} (qS_{\text{up}} + (1 - q) S_{\text{down}}) \]

Define

\[ X(t) \equiv \frac{S(t)}{e^{rt}} \]

therefore

\[ X_{\text{now}} = qX_{\text{up}} + (1 - q) X_{\text{down}} = \mathbb{E}_q (X_{\text{next}}) \]

which is a Martingale.

In general,

\[ S(0) = e^{-rT} \mathbb{E}_{RN} [S(T)] \]

then

\[ X(0) = \mathbb{E}_{RN} [X(T)] \]

Similarly

\[ f(0) = e^{-rT} \mathbb{E}_{RN} [f(T)] \]

Define

\[ Y(t) \equiv \frac{f(t)}{e^{rt}} \]

\[ \implies \]

\[ Y(0) = \mathbb{E}_{RN} [Y(T)] \]

In general

\[ \xi(t) = \mathbb{E}_Q [\xi(t)] , \quad t' > t \]

defines a Martingale.

Fundamental facts of financial derivatives:

1.

No Arbitrage \( \implies \exists Q \) — some probability measure

\[ \xi = \frac{\text{value of one option } f}{\text{value of another kind option } g} \]

then, \( \xi \) is a Martingale with respect to \( Q \).
2. No Arbitrage $\iff \exists$ (a Martingale measure)

3. 
   No Arbitrage + Completeness of the market $\iff \exists$ (! (a Martingale measure))

4. If $g$ is money market account, i.e., $g = e^{rt}$, then $Q$ is the risk-neutral probability.

Basics:
If 
$$
dy = \alpha(y,t) \, dt + \beta(y,t) \, dW
$$
then
$$
y \text{ is a martingale } \iff \alpha(y,t) \equiv 0
$$

Let us show if $\alpha(y,t) = 0$, then $y$ is a martingale. Since 
$$
dy = \beta(y,t) \, dW,
$$
\therefore \quad y(t) - y(0) = \int_0^t \beta(y,s) \, dW(s)
$$

$$
\mathbb{E}[y(t)] - y(0) = \mathbb{E} \int_0^t \beta(y,s) \, dW(s) = 0 \quad \text{(N.B. Ito Integral)}
$$
\therefore \quad y(0) = \mathbb{E}[y(t)]

1.2  Relationship between martingale and risk-neutral processes

We learned before that the risk-neutral process for the stock price movement is 
$$
dS = rSdt + \sigma SdW.
$$
Suppose 
$$
dS = \alpha dt + \beta dW
$$
if 
$$
\frac{S(t)}{e^{rt}} \text{ is a martingale } \implies \alpha = rS
$$
Proof:
$$
d \left( \frac{S(t)}{e^{rt}} \right) = d \left( S(t) e^{-rt} \right)
$$
$$
= e^{-rt} dS - re^{-rt} Sdt
$$
$$
= e^{-rt} (\alpha dt + \beta dW) - re^{-rt} Sdt
$$
$$
= e^{-rt} (\alpha - rS) dt + e^{-rt} \beta dW
$$
Therefore, 
\[ \alpha = rS \]
i.e., If \( S \) is a risk-neutral process, then \( e^{-rt}S(t) \) is a martingale.

### 1.3 Relationship between the BS PDE and Martingales

We have the following fact:

1. If \( V \) satisfies the BS PDE, then \( Ve^{-rt} \) is a martingale with respect to the risk-neutral measure.

2. No Arbitrage \( \implies Ve^{-rt} \) is a martingale in the risk-neutral measure.

Proof: (1)

\[
d \left( Ve^{-rt} \right) = e^{-rt}dV - re^{-rt}Vdt \\
= e^{-rt} \left( V_t dt + V_SdS + \frac{1}{2} \sigma^2 S^2 V_{SS}dt \right) - re^{-rt}Vdt
\]

Substituting the risk-neutral process:

\[
dS = rSdt + \sigma SdW
\]

leads to

\[
d \left( Ve^{-rt} \right) = e^{-rt} \left( V_t + \frac{1}{2} \sigma^2 S^2 V_{SS} + rSV_S - rV \right) dt + e^{-rt} \sigma SV_SdW
\]

\[= \text{the BS } \mathcal{L}_{BS} V, \therefore \text{it vanishes}
\]

\[= e^{-rt} \sigma SV_SdW
\]

or

\[
d \left( Ve^{-rt} \right) = \sigma S \left( Ve^{-rt} \right)_S dW
\]

i.e., the discounted \( V \) is a martingale.

(2)

No Arbitrage \( \implies V \) is a solution of BS PDE

\[ \implies \frac{V}{e^{rt}} \text{ is a martingale in the risk-neutral measure}
\]

\[ \therefore \frac{V(S(0),0)}{e^{-r \cdot 0}} = \mathbb{E}_{RN} \left[ \frac{V(S(T),T)}{e^{rT}} \right]
\]

\[V(S(0),0) = e^{-rT} \mathbb{E}_{RN} \left[ V(S(T),T) \right]
\]
2 The Market Price of Risk

Consider a natural process:
\[
\frac{d\theta}{\theta} = mdt + \Sigma dW
\]
where the drift term \( m \) is the expected growth rate and \( \Sigma \) is the volatility. \( W(t) \) is a Wiener process.

Note that \( \theta \) need not be the price of an investment asset. For example, it can be the noise level of Times Square or temperature in Beijing.

Suppose we write two derivatives \( V_1 \) and \( V_2 \) depending on only \( \theta \) and \( t \), i.e., some options or contracts that give a payoff as a function of \( \theta \) at some future time.

For simplicity, we assume there is no "dividend", i.e., no income before maturity.

Suppose
\[
\begin{align*}
\frac{dV_1}{V_1} &= \mu_1 dt + \sigma_1 dW \\
\frac{dV_2}{V_2} &= \mu_2 dt + \sigma_2 dW
\end{align*}
\]
where \( \mu_1, \mu_2, \sigma_1 \) and \( \sigma_2 \) are functions of time \( t \).

To eliminate risk, we construct the following portfolio:
\[
\Pi = \left( \sigma_2 V_2 \right) \times V_1 - \left( \sigma_1 V_1 \right) \times V_2
\]
where \( \sigma_2 V_2 \) units of \( V_1 \)
\[
= (\sigma_2 - \sigma_1) V_1 V_2
\]
\[
d\Pi = (\sigma_2 V_2) dV_1 - (\sigma_1 V_1) dV_2
\]
Substituting Eqs (1) and (2) into the above equation leads to
\[
d\Pi = (\sigma_2 V_2) V_1 (\mu_1 dt + \sigma_1 dW) - (\sigma_1 V_1) V_2 (\mu_2 dt + \sigma_2 dW)
\]
\[
= (\sigma_2 \mu_1 - \sigma_1 \mu_2) V_1 V_2 dt
\]
Since this portfolio is riskless, it must grows at risk-free rate, i.e.,
\[
d\Pi = r \Pi dt
\]
or
\[
(\sigma_2 \mu_1 - \sigma_1 \mu_2) V_1 V_2 = r (\sigma_2 - \sigma_1) V_1 V_2
\]
which can be rewritten as
\[
\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \equiv \lambda
\]
where \( \lambda \) is referred to as the market price of risk of \( \theta \). Note that
1. \( \lambda \) is a function of \( \theta \) and \( t \);

2. \( \lambda \) is independent of \( V_1 \) and \( V_2 \).

3. The ratio \( \frac{\mu - r}{\sigma} = \lambda \) is the same for all derivatives and it depends on \( \theta \) and \( t \) only. We can write

\[
\mu - r = \lambda \sigma
\]

Note that the LHS is the excess return above the risk-free rate, and \( \sigma \) is a measure of risk (or uncertainty), therefore, \( \lambda \) can be viewed the price of risk, i.e., the excess earning above the risk-free rate per unit \( \sigma \).

Now we can derive a PDE for any contingent claim \( V \) on \( \theta \) if

\[
\frac{dV}{V} = \mu dt + \sigma dW \tag{3}
\]

Using Ito’s Lemma, we have

\[
dV = \left( \frac{\partial V}{\partial t} + \theta m \frac{\partial V}{\partial \theta} + \frac{1}{2} \Sigma \theta^2 \frac{\partial^2 V}{\partial \theta^2} \right) dt + \theta \Sigma \frac{\partial V}{\partial \theta} dW \tag{4}
\]

Comparing Eq. (3) and Eq. (4) yields

\[
\mu V = \frac{\partial V}{\partial t} + \theta m \frac{\partial V}{\partial \theta} + \frac{1}{2} \Sigma \theta^2 \frac{\partial^2 V}{\partial \theta^2}
\]

\[
\sigma V = \theta \Sigma \frac{\partial V}{\partial \theta}
\]

or

\[
\mu = \frac{1}{V} \left[ \frac{\partial V}{\partial t} + \theta m \frac{\partial V}{\partial \theta} + \frac{1}{2} \Sigma \theta^2 \frac{\partial^2 V}{\partial \theta^2} \right]
\]

\[
\sigma = \frac{1}{V} \left( \theta \Sigma \frac{\partial V}{\partial \theta} \right)
\]

Since

\[
\mu - r = \lambda \sigma
\]

\[
\therefore \quad \frac{1}{V} \left[ \frac{\partial V}{\partial t} + \theta m \frac{\partial V}{\partial \theta} + \frac{1}{2} \Sigma \theta^2 \frac{\partial^2 V}{\partial \theta^2} \right] - r = \lambda \frac{1}{V} \left( \theta \Sigma \frac{\partial V}{\partial \theta} \right)
\]

i.e.,

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \Sigma \theta^2 \frac{\partial^2 V}{\partial \theta^2} + \theta (m - \lambda \Sigma) \frac{\partial V}{\partial \theta} - rV = 0 \tag{5}
\]

Note that
1. If $\theta$ is some stock price, i.e., a price of some investment asset, then $\theta$ can be viewed as a derivative too. Hence, it must satisfy the relation of market price of risk too:

$$\frac{m - r}{\Sigma} = \lambda$$

i.e.,

$$m - \lambda \Sigma = r$$

Substituting this into Eq. (5), we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2} \Sigma \theta^2 \frac{\partial^2 V}{\partial \theta^2} + \theta r \frac{\partial V}{\partial \theta} - rV = 0$$

which is the Black-Scholes equation!

2. In a risk-neutral world,

$$\lambda = 0$$

thus, e.g.,

$$\frac{dS}{S} = \mu dt + \sigma dW$$

and \( \frac{\mu - r}{\sigma} = 0 \)

$$\therefore \frac{dS}{S} = r dt + \sigma dW$$

Example: Bond options.

Stochastic spot rate (spot rate $\equiv$ short rate):

$$dr = m (r, t) \, dt + \Sigma (r, t) \, dW$$

(6)

e.g.,

1. Ho & Lee model

$$dr = \theta (t) \, dt + \sigma dW$$

2. Vasick/Hull-White model

$$dr = (\theta (t) - \alpha (t) r) \, dt + \sigma (t) \, dW$$

3. Cox-Ingersoll-Ross (CIR) model

$$dr = (\theta (t) - \alpha (t) r) \, dt + \sigma (t) \sqrt{r} \, dW$$

4. Black-Karasinski model

$$dr = r \left( \theta (t) - \frac{1}{2} \sigma^2 (t) - \alpha (t) \log r \right) \, dt + r \sigma (t) \, dW$$
Question: How to price a bond?

The issue is how to hedge — unlike a stock, one cannot go out and buy an interest rate, e.g., 10%.

The hedging strategy can be carried out as follows: Use two bonds with different maturities $T_1$ and $T_2$:

- Bond $V_1$: maturity $T_1$ 1 unit
- Bond $V_2$: maturity $T_2$ $-\Delta$ units

and the portfolio is

$$\Pi = (V_1 - \Delta V_2)$$

\[
d\Pi = \left( \frac{\partial V_1}{\partial t} dt + \frac{\partial V_1}{\partial r} dr + \frac{1}{2} \Sigma^2 \frac{\partial^2 V_1}{\partial r^2} dt \right)
- \Delta \left( \frac{\partial V_2}{\partial t} dt + \frac{\partial V_2}{\partial r} dr + \frac{1}{2} \Sigma^2 \frac{\partial^2 V_2}{\partial r^2} dt \right)
\]

By choosing

$$\Delta = \frac{\partial V_1}{\partial V_2}$$

to eliminate the random component, the no arbitrage argument leads to

\[
i.e., \left[ \frac{\partial V_1}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V_1}{\partial r^2} - \frac{\partial V_1}{\partial r} \left( \frac{\partial V_2}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V_2}{\partial r^2} \right) \right] = r \Pi dt
\]

\[
\implies \frac{\partial V_1}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V_1}{\partial r^2} - rV_1 = \frac{\partial V_1}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V_2}{\partial r^2} - rV_2
\]

Note that the LHS is a function of $T_1$ not $T_2$ while the RHS is a function of $T_2$ not $T_1$. Therefore, neither side depends on the maturity $T$, i.e.,

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V}{\partial r^2} - rV = \frac{\partial V}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V}{\partial r^2} - rV
\]

we can write

$$a (r, t) \equiv \Sigma (r, t) \lambda (r, t) - m (r, t)$$
where $\Sigma(r,t)$ and $m(r,t)$ are functions in Eq. (6). For this procedure to hold, we require

$$\Sigma(r,t) \neq 0.$$  

Therefore, the PDE for pricing a zero-coupon bond is

$$\frac{\partial V}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V}{\partial r^2} + (m - \lambda \Sigma) \frac{\partial V}{\partial r} - rV = 0$$

with final condition: $V(r,T) = \$1$

Note that

1. $\lambda = \lambda(r,t)$ is a function yet to be determined.

2. If we assume $V$ has the form:

$$V(r,t) = A(t,T) e^{-rB(t,T)}$$

then

$$\Sigma(r,t) = (\alpha(t)r - \beta(t))^{1/2}$$
$$m(r,t) = \left(-\gamma(t)r + \eta(t) + \lambda(r,t) [\alpha(t)r - \beta(t)]^{1/2}\right)$$

This leads to the following result:

$$m - \lambda \Sigma = -\gamma(t)r + \eta(t)$$

which is independent of $\lambda$!

3. Given those short-rate models above, we have, e.g.,

<table>
<thead>
<tr>
<th>Model</th>
<th>$\alpha$ Condition</th>
<th>$\beta$ Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vasick</td>
<td>$\alpha = 0$</td>
<td></td>
</tr>
<tr>
<td>CIR</td>
<td></td>
<td>$\beta = 0$</td>
</tr>
<tr>
<td>Hull-White</td>
<td>either $\alpha = 0$ or $\beta = 0$</td>
<td></td>
</tr>
</tbody>
</table>


3 Equivalent Martingale Measures

Recall: If $f$ and $g$ are some derivatives on the same, single process $dW$,

$$\text{No Arbitrage} \implies \frac{f}{g} \text{ is a martingale with respect to some measure}$$
More specifically,

\[ df = \mu_f f dt + \sigma_f f dW \]
\[ dg = \mu_g g dt + \sigma_g g dW \]

which are not necessarily geometric Brownian motions since \( \mu_f \) and \( \sigma_f \) can depend on \( f \), etc. An no arbitrage argument yields

\[ \frac{\mu_f - r}{\sigma_f} = \lambda = \frac{\mu_g - r}{\sigma_g} \]

For a new market price of risk \( \lambda^* \),

\[ \lambda^* = \frac{\mu^* - r}{\sigma} \]

we have

\[ df = (r + \lambda^* \sigma_f) f dt + \sigma_f f dW \] \hspace{1cm} (7)
\[ dg = (r + \lambda^* \sigma_g) g dt + \sigma_g g dW \] \hspace{1cm} (8)

Note that

1. Market price of risk determines the drift;
2. Volatility does not change;
3. Choosing a drift is equivalent to choosing a market price of risk \( \lambda \).

From the above argument, we conclude that:

No Arbitrage \( \Rightarrow \frac{f}{g} \) is a martingale for some \( \lambda \)

The question is which \( \lambda \). It turns out that

if \( \lambda = \sigma_g \),

then, \( \frac{f}{g} \) is a martingale for all derivative \( f \).

This can be demonstrated as follows:

Substituting \( \lambda = \sigma_g \) into Eqs. (7) and (8) yields

\[ df = (r + \sigma_g \sigma_f) f dt + \sigma_f f dW \]
\[ dg = (r + \sigma_g^2) g dt + \sigma_g g dW \]

Applying Ito's lemma to \( \ln f \) and \( \ln g \):

\[ d \ln f = \left( r + \sigma_g \sigma_f - \frac{1}{2} \sigma_f^2 \right) dt + \sigma_f dW \]
\[ d \ln g = \left( r + \frac{1}{2} \sigma_g^2 \right) dt + \sigma_g dW \]
therefore,

\[
\begin{align*}
  d \ln \frac{f}{g} &= d (\ln f - \ln g) \\
  &= d \ln f - d \ln g \\
  &= -\frac{1}{2} (\sigma_f - \sigma_g)^2 dt + (\sigma_f - \sigma_g) dW
\end{align*}
\]

Now, we want to use this to compute \( d \left( \frac{F}{g} \right) \) by the following method:

If we know

\[
\begin{align*}
  dX &= \mu_X dt + \sigma_X dW \\
  d\ln X &= \mu dt + \sigma dW
\end{align*}
\]

what is the relation between \((\mu_X, \sigma_X)\) and \((\mu, \sigma)\)? Since

\[
\begin{align*}
  d \ln X &= \frac{1}{X} dX + \frac{1}{2} \sigma_X^2 \left( -\frac{1}{X^2} \right) dt \\
  &= \left( \frac{1}{X} \mu_X - \frac{1}{2} \sigma_X \frac{1}{X^2} \right) dt + \frac{\sigma_X}{X} dW \\
  \therefore \\
  \mu &= \frac{1}{X} \mu_X - \frac{1}{2} \sigma_X \frac{1}{X^2} \\
  \sigma &= \frac{\sigma_X}{X}
\end{align*}
\]

i.e.,

\[
\begin{align*}
  \sigma_X &= \sigma X \\
  \mu_X &= \left( \mu + \frac{1}{2} \sigma^2 \right) X
\end{align*}
\]

therefore,

\[
\begin{align*}
  d \left( \frac{F}{g} \right) &= \left[ -\frac{1}{2} (\sigma_f - \sigma_g)^2 + \frac{1}{2} (\sigma_f - \sigma_g)^2 \right] dt + (\sigma_f - \sigma_g) \frac{f}{g} dW \\
  \therefore \\
  d \left( \frac{F}{g} \right) &= (\sigma_f - \sigma_g) \frac{f}{g} dW
\end{align*}
\]

in which there is no drift term, thus, \( \frac{F}{g} \) is a martingale.

Note that

1. When the market price of risk = \( \sigma_g \), it is a world of forward risk-neutral with respect to \( g \).
2. Since $\frac{f}{g}$ is a martingale,

\[
\therefore \quad \frac{f_0}{g_0} = \mathbb{E}_g \left[ \frac{f_T}{g_T} \right] \\
\implies \quad f_0 = g_0 \mathbb{E}_g \left[ \frac{f_T}{g_T} \right]
\]

(a) Money Market Account as the Numeraire:

Since

\[ dg = rg \, dt \]

where $r$ can be stochastic, but the volatility of $g = 0$, i.e., the market price of risk is zero for the money market. Then

\[ f_0 = g_0 \mathbb{E}_{RN} \left[ \frac{f_T}{g_T} \right] \]

Since

\[ g_0 = 1 \]
\[ g_T = e^{\int_0^T r(\tau) \, d\tau} \]

\[ \therefore \quad f_0 = \mathbb{E}_{RN} \left[ e^{-\int_0^T r(\tau) \, d\tau} f_T \right] \]

If $r$ is a constant, then

\[ f_0 = e^{-rT} \mathbb{E}_{RN} \left[ f_T \right] \]

Hence, the money market account numeraire is equivalent to traditional risk-neutral world.

(b) Zero-Coupon bond price as the Numeraire:

Definition: $B(t,T)$ is the price at time $t$ of a zero-coupon bond worth of $\$1$ at time $T$.

Then,

\[ g_T = B(T,T) = 1 \]
\[ g_0 = B(0,T) \]

\[ \therefore \quad f_0 = B(0,T) \mathbb{E}_T [ f_T ] \]

where $\mathbb{E}_T$ denotes the forward risk-neutral measure with respect to $B(t,T)$. Note that it is nice to have $B(0,T)$ outside $\mathbb{E}$-operator.

Recall the forward price of $f$ maturing at $T$ is

\[ F = \frac{f_0}{B(0,T)} \]
e.g., \( F = S_0e^{rT} \). Since
\[
\begin{align*}
 f_0 &= B(0,T) \mathbb{E}_T[f_T] \\
 F &= \frac{f_0}{B(0,T)} \\
 \therefore F &= \mathbb{E}_T[f_T]
\end{align*}
\]
i.e., In a forward risk-neutral measure with respect to \( B(0,T) \), the forward price of \( f \) is equal to the expected future spot price. In contrast, in the traditional risk-neutral measure, futures price is equal to the expected future spot price.

Intuitive way (via binomial trees) of understanding change of numeraire:

Since
\[
 f_{\text{now}} = e^{-r\delta t} (q f_{\text{up}} + (1 - q) f_{\text{down}})
\]
where \( q \) is the risk-neutral probability depending on the underlying movement. For another tradeable \( g \), we have
\[
 g_{\text{now}} = e^{-r\delta t} (q g_{\text{up}} + (1 - q) g_{\text{down}})
\]
therefore
\[
\frac{f_{\text{now}}}{g_{\text{now}}} = \frac{e^{-r\delta t} (q f_{\text{up}} + (1 - q) f_{\text{down}})}{e^{-r\delta t} (q g_{\text{up}} + (1 - q) g_{\text{down}})} = \frac{q g_{\text{up}} f_{\text{up}} + (1 - q) g_{\text{down}} f_{\text{down}}}{q g_{\text{up}} + (1 - q) g_{\text{down}} g_{\text{up}}}
\]

If
\[
 q^* \equiv \frac{q g_{\text{up}}}{q g_{\text{up}} + (1 - q) g_{\text{down}}}
\]
Since
\[
\frac{(1 - q) g_{\text{down}}}{q g_{\text{up}} + (1 - q) g_{\text{down}}} + q^* = \frac{(1 - q) g_{\text{down}}}{q g_{\text{up}} + (1 - q) g_{\text{down}}} + \frac{q g_{\text{up}}}{q g_{\text{up}} + (1 - q) g_{\text{down}}}
\]
\[
= \frac{(1 - q) g_{\text{down}} + q g_{\text{up}}}{q g_{\text{up}} + (1 - q) g_{\text{down}}} = 1
\]
i.e.,
\[
\frac{(1 - q) g_{\text{down}}}{q g_{\text{up}} + (1 - q) g_{\text{down}}} = 1 - q^*
\]
therefore,
\[
\frac{f_{\text{now}}}{g_{\text{now}}} = q^* \left( \frac{f_{\text{up}}}{g_{\text{up}}} \right) + (1 - q) \left( \frac{f_{\text{down}}}{g_{\text{down}}} \right)
\]
Of course, $q^*$ will vary from subtree to subtree. Therefore,

$$\frac{f_{\text{now}}}{g_{\text{now}}} = E_* \left[ \frac{f_{\text{next}}}{g_{\text{next}}} \right]$$

where $*$ denotes the expectation with respect to $q^*$. Iterate through the tree, we have

$$\frac{f(t)}{g(t)} = E_* \left[ \frac{f(T)}{g(T)} \right]$$

therefore,

$$\frac{f}{g} \text{ is a martingale with respect to } q^*$$

i.e.,

$$f(t) = g(t) E_* \left[ \frac{f(T)}{g(T)} \right]$$