Problem Set 1

G63.2707 Fall 2002
Modern Statistical Inference And Econometrics Due September 23

For problems 1 through 4, find the maximum likelihood (ML) estimate \( \hat{\theta} \) for the parameter \( \theta \) given observations \( x_1, \ldots, x_n \) on i.i.d. random variables with the indicated distribution on the indicated subset of \( \mathbb{R} \).

1. The distribution with mass function
   \[
   p(x|\theta) = \frac{e^{-\theta} \theta^x}{x!} \quad \text{for } x = 0, 1, 2, \ldots
   \]
   The parameter \( \theta \) must be positive. Assume that not all the \( x_i \) are zero.

2. The distribution with density function
   \[
   f(x|\theta) = (1 + \theta)x^\theta \quad \text{for } 0 < x < 1.
   \]
   The parameter \( \theta \) must be \( > -1 \).

3. The distribution with density function
   \[
   f(x|\theta) = \frac{1}{2}\theta^3x^2e^{-\theta x} \quad \text{for } x > 0.
   \]
   The parameter \( \theta \) must be positive.

4. The distribution with density function
   \[
   f(x|\theta) = \frac{1}{2}e^{-|x-\theta|} \quad \text{for } -\infty < x < \infty.
   \]

5. Suppose \( x_1, \ldots, x_n \) are a random sample from a Cauchy distribution, having density
   \[
   f(x|\theta, \omega) = \frac{1}{\pi \omega} \cdot \frac{1}{1 + (\frac{x-\theta}{\omega})^2} \quad \text{for } -\infty < x < \infty.
   \]
   Here \( \theta \) can be any real number, but \( \omega \) must be positive.

   (a) Find the logarithm of the likelihood function \( L \) for the parameters \( \theta \) and \( \omega \).
(b) Differentiate log \( L \) with respect to \( \theta \) and \( \omega \) to find the first-order conditions satisfied by the ML estimates \( \hat{\theta} \) and \( \hat{\omega} \).

(c) Suppose the actual data are these twenty observations:

\[
\begin{array}{ccccccc}
282.6 & 313.9 & 297.8 & 288.1 & 92.2 \\
289.5 & 284.6 & 276.6 & 287.0 & 286.5 \\
280.4 & 268.0 & 295.2 & 296.5 & 292.3 \\
303.6 & 280.1 & 261.6 & 280.2 & 317.9 \\
\end{array}
\]

[Note: For those who prefer to copy and paste rather than retype, these data are also in a text file that you can access by clicking “Data for problem 5(c) in Problem Set 1” on the course web page.] By maximizing log \( L \) directly (or by finding solutions to the first-order conditions if you prefer), find \( \hat{\theta} \) and \( \hat{\omega} \).

6. (Non-parametric maximum likelihood.) Let \( x_1, \ldots, x_n \) be observations on i.i.d. random variables having some unknown distribution \( F \) on \( \mathbb{R}^m \). For simplicity, assume that the observations are distinct; i.e. \( x_i \neq x_j \) if \( i \neq j \). Except for the observed data, absolutely nothing is known or assumed about \( F \). Show that the ML estimate \( \hat{F} \) for \( F \) is the discrete distribution that has mass \( 1/n \) at each \( x_i \). [Note: This discrete distribution is called the empirical distribution, denoted by \( F_n \).]

7. This exercise asks you to carry out a simple simulation to compare the sampling distributions of the mean and the median, for (pseudo-) random samples of size \( n = 15 \) generated from a normal distribution. Let \( J \), the number of “trials” of the simulation, equal 10,000. For \( j = 1, \ldots, J \), generate random numbers \( x_1^{(j)}, \ldots, x_{15}^{(j)} \) from the standard normal distribution \( N(0, 1) \), and compute the mean \( \bar{x}_j \) and the median \( m_j \). Finally, compute and compare

\[
SD_{\text{mean}} = \sqrt{\frac{1}{J} \sum_{j=1}^{J} \bar{x}_j^2}
\]

and

\[
SD_{\text{median}} = \sqrt{\frac{1}{J} \sum_{j=1}^{J} m_j^2}
\]

What does the comparison suggest about the desirability of the sample mean relative to the median, as a way to estimate the center of a normal distribution?
8. This exercise applies an alternative to ML, called the *method of moments*, to the estimation of the parameter $\theta$ in problem 2.

(a) For the density $f(x|\theta) = (1 + \theta)x^\theta$ on the set $0 < x < 1$, find the expected value of $x$ as a function of the parameter $\theta$. Denote this expected value by $\mu(\theta)$.

(b) The method-of-moments estimator $\tilde{\theta}$ is defined as the solution to

$$
\mu(\tilde{\theta}) = \bar{x}
$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$. In other words, we "set the population moment equal to the sample moment." Solve for $\tilde{\theta}$ (as a function of $\bar{x}$). Notice that $\tilde{\theta}$ differs from the ML estimate $\hat{\theta}$. 
