Homework 9

Objective: Newton’s iteration to solve nonlinear equations. Numerical solution of nonlinear optimization problems.

1. In order to find the square root of a positive number \( a > 0 \), solve the nonlinear equation \( f(x) = x^2 - A = 0 \) with Newton’s iteration.

   (i) Write down Newton’s iteration for this function.

   (ii) Implement it with a Matlab function \( \text{root} = \text{mysqrt}(A, \text{eps0}) \). Do not provide your code, but do have a suitable stop criterion built in the function which guarantees double precision \( \text{eps0} = 2.0 \cdot 10^{-16} \). Provide your stop criterion.

   (iii) Choose your own initial guess to find roots for the three cases \( A = 3 \), \( A = 1.0 \cdot 10^4 \cdot \pi \), and \( A = \exp(-200) \). Note that you must first compute \( A \) according to these three formulae and then send each of them to the function \( \text{mysqrt}(A, \text{eps0}) \). For each root finding process, show the relative errors for every Newton iteration (as compared to \( \text{rootexact} = \sqrt{A} \), which by itself is not exact, it is accurate to 16 digits).

2. The Schult’s method for inverting an \( n \times n \) matrix \( A \) is described in Problem 5.28, page 254. Do part (a) of this problem; namely, prove

\[
R_{k+1} = R_k^2, \quad E_{k+1} = E_k A E_k
\]

(1)

3. Denote by \( J_0(z) \) the Bessel J function of order 0; in Matlab this function is computed by \( \text{nu}=0; \text{bj}=\text{besselj(nu,z)} \). Find three real numbers \( a, b, c \) to minimize the quantity

\[
F(a, b, c) =: \int_a^\beta \left[ J_0(z) - a \cos(b z + c) \right]^2 \, dz, \quad \alpha = 0, \quad \beta = 2\pi
\]

(2)

using Newton’s iteration. In order to numerically solve this problem (which involve an integral which we don’t know how to do analytically), we must discretize the integral: To replace it with a finite sum. This can be done before writing down the Newton’s iteration formula or after it. We'll discretize before it; namely we use \( n = 40 \) equispaced points

\[
\{ z_i = \alpha + i h \mid h = (\beta - \alpha)/n, \ i = 1, 2, \ldots, n \}\)

(3)

to rewrite the original problem (2) as

\[
\min f(a, b, c) =: h \sum_{i=1}^n \left[ J_0(z_i) - a \cos(b z_i + c) \right]^2,
\]

(4)

a. Plot the function \( J_0(z) \) in \([\alpha, \beta]\).

b. Find \( a, b, c \) to solve the optimization problem (4) by finding the roots of the gradient of \( f \). Namely, solve the three equations \( \nabla f(a, b, c) = 0 \) by Newton’s iteration (denote a root by \((\hat{a}, \hat{b}, \hat{c})\), a point in 3-D)

c. Provide the minimum value \( f \) and the gradient \( \nabla f \) at \((\hat{a}, \hat{b}, \hat{c})\). For this \((\hat{a}, \hat{b}, \hat{c})\), plot the error function

\[
e(z) = J_0(z) - u(z), \quad \text{where} \quad u(z) = \hat{a} \cos(\hat{b} z + \hat{c}).
\]

(5)

d. Find as many solutions (local minima) as you can, and for each of them do part (c). Which one gives rise to the global minimum of our problem?