Homework 3, Part II

Objective: Least-squares solution of under-determined linear system with the “physically relevant norms”. Optimal design of narrow and wide bandwidth digital filters and high order quadrature formulae.

A filter $f$ is required to be a smooth transition from 0 to 1, to be monotone ($f'(x) \geq 0$ for $x \in (a,b)$), and to have vanishing derivatives up to certain order at the two end points $a = -\pi$, $b = \pi$. If done correctly, the function

$$ f_3(x) = \frac{1}{2\pi}(x + \pi) + \sum_{j=1}^{n} c_j \sin(jx) \quad (1) $$

will satisfy the requirements (except $f' \geq 0$) with suitably chosen coefficients $c_j$. In Homework 2, you have set up an m-by-n linear system for $c_j$, solved it with $m = n = 5$ and got a solution which gives rise to an $f_3$ that happens to be monotone.

There is no reason that we must choose $m = n$ and get a square linear system to which the solution is unique. On the contrary, we prefer $m < n$ so that the linear system is under-determined: there are more degrees of freedom than the number of equations. When this happens, the solution is not unique, and therefore there is freedom to choose a solution with desirable properties.

**Question 1.** Let $D \in \mathbb{R}^{n \times n}$ be invertible. Describe with no more than 40 words (plus necessary formulae) a procedure which uses SVD to solve our problem $A \cdot c = b$ and simultaneously minimizes the 2-norm of $x = D \cdot c$. Hint: Consider the equation $A \cdot D^{-1} \cdot D \cdot c = b$. Note: for this problem to make practical sense, it is assumed that $m < n$ (what happens if $m = n$).

**Question 2.** Find out $D$ if we want to solve our problem $A \cdot c = b$ and minimize the 2-norm of $f_3'$, go to the end of this handout for details on the 2-norm of a continuous function.

**Question 3.** For $m = 5$, $n = 8$, solve our problem $A \cdot c = b$ and minimize the 2-norm of $f_3^{(2m+2)}$. Show the condition number. Plot $f_3$ so constructed with 32 equispaced points in $[-\pi, \pi]$.

**Question 4.** For $m = 5$, $n = 31$, $\mu = 0.01$, solve our problem $A \cdot c = b$ and minimize $\|f_3^{(4)}\|_2^2 + \mu\|f_3^{(4)}\|_2^2$. Show the condition number. Plot $f_3$ so constructed with 80 equispaced points in $[-\pi, \pi]$.

**Appendix** The 2-norm of a function $f$ in $[a,b]$ is defined by the formula

$$ \|f\|_2 = \left( \frac{1}{b-a} \int_a^b f^2(x)dx \right)^{\frac{1}{2}}; \quad (2) $$

therefore it can verified that a sine or cosine series

$$ f(x) = \sum_{j=1}^{n} a_j \sin(jx), \quad \text{or} \quad f(x) = \frac{1}{2}a_0 + \sum_{j=1}^{n} a_j \cos(jx) \quad (3) $$

in $[a,b] = [-\pi, \pi]$ will have the 2-norm (up to a constant multiple)

$$ \|f\|_2 = \left( \sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}}, \quad \text{or} \quad \|f\|_2 = \left( \sum_{j=0}^{n} |a_j|^2 \right)^{\frac{1}{2}} \quad (4) $$