Third Assignment (Due Wednesday February 18)

The effect of a fixed point source of light with frequency $\omega$ in three dimensional space can be modeled by the forced wave equation

$$u_{tt} - \Delta u = \delta(x) e^{i\omega t},$$  \hspace{1cm} (1)

where the real part of $u$ determines the physical solution. If the source is moving with constant speed $v$, on the other hand, the equation becomes

$$u_{tt} - \Delta u = \delta(x - vt) e^{i\omega t}.$$ \hspace{1cm} (2)

The goal of this exercise is to compare the solutions to these two equations (i.e., to observe the non–relativistic Doppler effect). In principle, both equations can be solved using Duhamel’s principle. Yet dealing with the $\delta$’s is a subtle matter, so we’ll make sure everything’s going fine by first solving the problem with a fixed source in a simpler fashion, then using this solution to check the one through Duhamel’s principle, and finally changing the auxiliary function $u(x, t, s)$ in Duhamel’s principle to account for a moving source.

a) For equation (1), propose a solution of the form

$$u(x, t) = p(x)e^{i\omega t}.$$  

The resulting elliptic equation, denoted the reduced wave equation, is

$$\Delta p + \omega^2 p = -\delta(x)$$

Thus we are looking for a fundamental solution to the reduced wave operator. Find such solution, using radial symmetry, and proposing a solution of the form

$$p = c \frac{e^{i\lambda r}}{r}$$

where $\lambda$ and $c$ are constants.

b) Now solve again equation (1), this time using Duhamel’s principle, and check your solution with that of part a). Determine $u(x, t, s)$.

c) Clearly, the auxiliary functions $u(x, t, s)$ for equations (1) and (2) are related, since for each value of $s$, the corresponding auxiliary problems are the same, only displaced in $x$. Thus write $u(x, t, s)$ for equation (2), and find $u(x, t)$ (you may assume that $v < 1$: the source moves at a slower speed than light).

d) Finally, compare the two solutions. In particular, you may want to look at the solution to (2) at a fixed point (i.e., take the position of an observer looking at the moving signal). Is the frequency of the received signal a constant? Try to argue a qualitative reason for the observed behavior.