A) **Complete the Square:** Solve the Laplace equation on the right triangle formed by the lower right corner of the unit square ($0 < y < x < 1$). The boundary conditions are

\[
\begin{align*}
    u &= 0 & \text{on the diagonal (} x = y \text{)} \\
    u &= f(x) & \text{on the } x\text{-axis (} y = 0 \text{)} \\
    u &= 0 & \text{on the right-side (} x = 1 \text{)}
\end{align*}
\]

For an interesting enough $f(x)$, plot the solution in Matlab. How do the continuity properties of the solution depend on $f(x)$?

B) **A Wave Beacon:** Consider a beacon constructed by applying a sinusoidal voltage potential to two (separated) half-circles.

![Wave Beacon Diagram]

The time-harmonic external wave field $w(r, \theta, t)$ satisfies the two-dimensional wave equation

\[
w_{tt} - \nabla^2 w = 0
\]

on $1 < r < +\infty$

with BC: $w(1, \theta, t) = e^{-it} \begin{cases} -1 & \text{for } \pi/2 < \theta < 3\pi/2 \text{ (leftside)} \\
+1 & \text{otherwise (rightside)} \end{cases}$

and far-field out-going waves.

Obtain a series representation of the solution and make a Matlab movie or plot sequence showing the wave radiation (real part). Also show that choosing the “wrong” Hankel function gives a very different picture (make sure the BC at $r = 1$ is satisfied). Produce an estimate for the angular plot of the beacon’s directionality

\[
D(\theta) = \lim_{r \to \infty} \left( r |w(r, \theta, t)|^2 \right)
\]

and interpret the results.

C) **Sounds of the City:** As a warm-up, show that the general form of spherically symmetric solutions to the wave equation in three space dimensions has the form

\[
u(r, t) = \frac{f(r - ct) + g(r + ct)}{r}
\]
• It is then not surprising that the free-space (and out-going) Green’s function for the wave equation has the same form

$$G(x, y, z, t; \tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = \frac{1}{4\pi c} \frac{\delta[\tilde{r} - c(t - \tilde{t})]}{\tilde{r}}$$  \hspace{1cm} (1)$$

where $\tilde{r}^2 = (x - \tilde{x})^2 + (y - \tilde{y})^2 + (z - \tilde{z})^2$ (This is discussed at the very end of Section 10-5 and in Section 10-7 in Guenther & Lee.) This Green’s function (with $c = 1$) satisfies

$$G_{tt} - \nabla^2 G = \delta(x - \tilde{x}, y - \tilde{y}, z - \tilde{z}, t - \tilde{t})$$

on $-\infty < x, y, z, t < +\infty$

which gives the interpretation of (1) as the response to a point forcing at the location $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$.

• Fill in the details in the following derivation of a moving source problem for the three-dimensional wave equation. The forcing by a history of point sources $(\tilde{t} < t)$ that moves along a trajectory $(\tilde{x}(\tilde{t}), \tilde{y}(\tilde{t}), \tilde{z}(\tilde{t}))$ with strength $A(\tilde{t})$ can be described by the (superposition) integral

$$F(x, y, z, t) = \int_{-\infty}^{t} A(\tilde{t}) \delta[x - \tilde{x}(\tilde{t}), y - \tilde{y}(\tilde{t}), z - \tilde{z}(\tilde{t}), t - \tilde{t}] \, d\tilde{t}.$$

For a point source moving along the $x$-axis with uniform velocity $(0 < a < 1)$ with a time-oscillatory amplitude corresponding to

$$\tilde{x}(\tilde{t}) = a\tilde{t} \ ; \ \tilde{y}(\tilde{t}) = 0 \ ; \ \tilde{z}(\tilde{t}) = 0 \ ; \ A(\tilde{t}) = \sin(\tilde{t}) \ .$$

The wave PDE for this situation is given by

$$v_{tt} - \nabla^2 v = F(x, y, z, t)$$

on $-\infty < x, y, z, t < +\infty$.

The delta function in the integral allows for explicit evaluation (there is an important geometric aspect to this step), give this expression for $v(0, 1, 0, t)$.

Plot the solution using Matlab and interpret the solution.

![Graph](image-url)