Mathematics of Finance Final Exam  
Fall 2004 – Professor Kohn  
December 20, 2004

• This is a closed-book exam. You may use two sheets of notes (8.5 × 11, both sides, any font). You may use a calculator if you wish, however you do not need one. Answers may be left in any reasonably simple correct form, for example \( \sqrt{2} \) is just as good as 1.414.

• Put your answers on the exam paper; use the back of the page if you need more space, and attach additional sheets if necessary. I will grade only your exam paper, not your scratch paper.

• Part A consists of 10 “shorter-answer” problems, worth 10 points each. Part B consists of 3 “longer-answer” problems, worth 20 each. The total possible score is thus 160.

• Show your work, and explain all answers (at least briefly). Partial credit will be given for correct ideas.

NAME: ______________________

A) Shorter-answer problems: 10 points each

A1) _____  A6) _____  
A2) _____  A7) _____  
A3) _____  A8) _____  
A4) _____  A9) _____  
A5) _____  A10) _____

B) Longer-answer problems: 20 points each

B1) _____  
B2) _____  
B3) _____

Total: __________
Part A: Shorter-answer questions. The problems in Part A can be answered relatively briefly; they are worth 10 points each.

1. (10 points) True or false: “in mean-variance analysis, there is no limit to the mean return you can achieve, provided you’re willing to accept a large enough risk.” (You must justify your answer to get credit.)

2. (10 points) Consider an asset whose daily return has mean $r_d$ and standard deviation $\sigma_d$. Assume distinct days are independent, ignore compounding, and use a 365-day year, so the annual return is $r_{\text{ann}} = r_d^{(1)} + \cdots + r_d^{(365)}$. If the mean annual return is $\overline{r}_{\text{ann}} = 10\%$ and the standard deviation of the annual return is $\sigma_{\text{ann}} = 20\%$, what are $r_d$ and $\sigma_d$?
3. (10 points) Consider the following multiperiod investment problem. There are two risky assets, Asset 1 and Asset 2. Each doubles or halves with probability 1/2, and they are independent. A multiperiod investor pursues the strategy of placing fraction $\alpha_1$ of his wealth in Asset 1, fraction $\alpha_2$ of his wealth in Asset 2, and fraction $1 - \alpha_1 - \alpha_2$ in his pocket at each time period. Characterize the random variable which gives his single-period return.

4. (10 points) Explain why “the expected return is not to be expected.”
5. (10 points) What portfolio of calls has the payoff diagram shown below?

[In words: the payoff $f(s_T)$ is piecewise linear, vanishing for $s_T < 10$ and $s_T > 18$. The slope is discontinuous at $s_T = 10, 12, 14, 16, 18$ and the payoffs there are $f(10) = 0, f(12) = 2, f(14) = -2, f(16) = 2,$ and $f(18) = 0$.]

6. (10 points) The current price of XYZ stock is $s_0 = 10$. The risk-free rate is $r = 0$. Call options with a one-year maturity and strike price 10 can be bought and sold for 5. Put options with the same strike and maturity can be bought and sold for 4. Show that put-call parity is violated. Specify what purchases and sales you should execute to take make a risk-free profit by taking advantage of this imbalance.
7. (10 points) Our friend Victor is thinking again of buying some lottery tickets. Suppose there are $10^6$ tickets. There is just one prize-winning ticket, worth $10^5$ dollars; the other tickets win nothing. Victor’s current wealth is $10^5$; each ticket costs 1 dollar, and he plans to buy $n$ tickets. Characterize the random variable which gives his wealth after the lottery.

8. (10 points) Consider an asset with lognormal return, i.e. whose return has the form $R = e^X$ where $X$ is Gaussian with mean $m$ and standard deviation $\sigma$. What is the probability that $R < E[R]$?
9. (10 points) On a binomial tree we value options by working backward, using the basic formula \( f_{\text{now}} = e^{-r\delta t}[q f_{\text{up}} + (1-q) f_{\text{down}}] \) where \( q = (e^{r\delta t}s_{\text{now}} - s_{\text{down}})/(s_{\text{up}} - s_{\text{down}}) \). Explain briefly the origin of this formula.

10. (10 points) The continuous-time Black-Scholes theory says the value of an option with payoff \( f(s_T) \) is \( e^{-rT}E[f(s_0 e^X)] \) where \( X \) is Gaussian with mean \((r - \frac{1}{2}\sigma^2)T\) and variance \( \sigma^2 T \). Using this, give a formula for the value of the option whose payoff is \( f(s_T) = \ln s_T \).
Part B: Longer-answer questions. The problems in part B are worth 20 points, because they have several parts.

1. (20 points) Consider mean-variance analysis for a market with three risky assets, whose returns are \( r_1, r_2, \) and \( r_3. \) The covariance matrix and mean returns are

\[
V = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \varphi = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.
\]

You may find it useful to know that

\[
V^{-1} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 1 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}.
\]

(a) Find the weights of the minimum-variance portfolio.

(b) Find the weights of a second mean-variance-efficient portfolio.

(c) Describe how the weights of the most general mean-variance-efficient portfolio are related to your answers to (a) and (b). Which ones maximize (rather than minimize) the mean return, for given variance?
2. (20 points) Suppose stock prices are restricted to the following two-period tree.

\[ \text{In words: } s_0 = 50; \text{ at the intermediate time } t = \delta t \text{ the price is either } 60 \text{ or } 40; \text{ at the final time } t = 2\delta t \text{ the price is either } 70, 50, \text{ or } 30. \] The risk-free rate is \( r = 0 \).

Consider a call option with strike price 40 which matures at the final time.

(a) Use the tree to value this option.

(b) An investment bank selling this option wishes to avoid any risk. How much should it charge for the option, and what should it do to create a replicating portfolio at the initial time period?

(c) Suppose the stock goes down to 40 at the initial time period. Exactly how much stock should the investment bank buy or sell at this time?
3. (20 points) Consider the multiperiod binomial model in which a stock’s price is initially $s_0$, and at each time period it increases by a factor of $u$ or decreases by a factor of $d$. Suppose the probability of going up at any time step is $p$, and distinct times are independent. Consider the situation after 1000 time steps.

(a) Express the final price in terms of $s_0, d, u,$ and the total number $Y$ of times the stock went up.

(b) Find the mean and variance of $Y$.

(c) The central limit theorem says $Y$ is nearly Gaussian. It follows that the logarithm of the final stock price after 1000 steps is nearly Gaussian. Find its mean $m$ and standard deviation $\sigma$ in terms of $u, d,$ and $p$. 