Mathematics of Finance – Notes on Options, Part 3 – 12/1/03

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This material is extracted (with some editing and additions) from my Derivative Securities lecture notes (available in full at www.math.nyu.edu/faculty/kohn). Much of it is in Luenberger but I like my way of presenting it better. Other good books providing more in-depth treatments include J.C. Hull, Options, futures, and other derivatives, R. Jarrow and S. Turnbull, Derivative securities, and N. Chriss, Black Scholes and beyond.

Recap of the binomial tree option pricing formula. Let’s review the main accomplishments of Part 2. For a one-period binomial tree we obtained the pricing formula

\[ f_{\text{now}} = e^{-r\delta t}[q f_{\text{up}} + (1-q)f_{\text{down}}] \]

where

\[ q = \frac{e^{r\delta t}s_{\text{now}} - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}} \]  \hspace{1cm} (1)

We can write the first equation as

\[ \text{option value} = e^{-rT}E_{\text{RN}}[f(s_T)] \]

where \( T = \delta t \) is the maturity time (since we’re considering just one time period), \( f(s_T) \) is the option payoff at maturity (which can take just two possible values, \( f(s_{\text{up}}) = f_{\text{up}} \) or \( f(s_{\text{down}}) = f_{\text{down}} \) for a one-period binomial tree), and \( E_{\text{RN}} \) denotes the expected value using the biased coin with probability \( q \) of the up state and \( 1-q \) of the down state. We call this the “risk-neutral” probability distribution, for reasons explained below.

For a multiperiod multiplicative binomial tree we obtained a pricing formula that looks more complicated, but is in a certain sense the same. Indeed, we found that

\[ \text{option value now} = e^{-rT}\sum_{j=0}^{N} \binom{N}{j} q^{j}(1-q)^{N-j} f(s_0 u^j d^{N-j}) \]  \hspace{1cm} (2)

where \( T = N\delta t \) is the maturity time, \( s_0 \) is the stock price now (at time 0), and

\[ q = \frac{e^{r\delta t} - d}{u - d}. \]

The formula for \( q \) is consistent with (1) since we are now assuming \( s_{\text{up}} = us_{\text{now}} \) and \( s_{\text{down}} = ds_{\text{now}} \) throughout the tree. Formula (2) can again be written as

\[ \text{option value} = e^{-rT}E_{\text{RN}}[f(s_T)] \]

where \( E_{\text{RN}}[f(S_T)] \) is the expected final-time payoff associated with the “risk-neutral probability distribution,” i.e. the probability associated with an independent coin-flip at each time period, with probability \( q \) of going up and \( 1 - q \) of going down.
Our discussions of hedging show that if the stock price is truly confined to the tree then these option prices are forced upon us. Indeed, an investment bank selling the option and pursuing the right trading strategy (holding at each stage the “replicating portfolio”) takes no risk. If the option were available in the market at any other price, there would be an arbitrage opportunity (i.e. an opportunity to profit without taking any risk).

Why is this surprising? Our analysis shows that the value of an option is its “discounted risk-neutral expected payoff.” Before the Black-Scholes-Merton theory people thought that selling or buying an option involved risk. The common view was that each investor could assess whether the option was an attractive investment (at a given price) by considering the expected final-time utility of its payoff.

The work of Black, Scholes, and Merton showed that this is wrong – at least if the underlying has lognormal dynamics. An option can be replicated by an appropriate trading strategy. So its value is determined by arbitrage: the option’s value is the (initial) cost of the replicating portfolio. The situation is just like pricing a forward contract – except that the value of a forward is independent of the underlying dynamics (since no trading is required) whereas the value of an option depends on the underlying dynamics (typically lognormal, or a binomial tree approximation thereof).

We’ve shown that the option’s value is $e^{-rT} E_{RN}[f(s_T)]$, the discounted final-time expected payoff calculated using the risk-neutral probability. On a tree, you evaluate the risk-neutral expectation $E_{RN}[f(s_T)]$ by pretending the stock goes up with probability $q$ and down with probability $1 - q$ at each timestep, and taking the expected final-time payoff. Notice that we use the “risk-neutral” probability of the up state, $q$, not the real probability of the up state (which for the standard binomial approximation of lognormal dynamics would be $p = \frac{1}{2}(1 + \mu \sqrt{\Delta t})$). Later in these notes we’ll discuss the case of continuous time. We’ll explain that one evaluates evaluate $E_{RN}[f(s_T)]$ by pretending $s_T$ is lognormal with drift $r - \frac{1}{2}\sigma^2$ and volatility $\sigma$, and taking the expected final-time payoff. Again, one must use the “risk-neutral” drift $\mu_{RN} = r - \frac{1}{2}\sigma^2$ rather than the real drift $\mu$.

The term “risk-neutral” is just an analogy, i.e. a way to think about and remember this formula. Recall that utility functions are always concave, and an investor’s risk-averseness is reflected in the degree of concavity. There is no such thing as a risk-neutral investor; but if there were, his/her utility function would be linear rather than concave. Constant factors are irrelevant, so the risk-neutral utility $U$ is the trivial function $U(w) = w$. For this $U$, the expected final-time utility of the option’s payoff is $E[f(s_T)]$. If in addition we use the proper probability distribution (the “risk-neutral” one, not the real one) then we get the option’s value, up to the discount factor $e^{-rT}$.

OK, but why does the valuation formula take this form? The multiperiod valuation formula is obtained by iterating the single-period version, so the essential question is this:
Why, for a one-period binomial tree, does an option with payoffs $f_{\text{up}}$ and $f_{\text{down}}$ have value

$$f_{\text{now}} = e^{-r\Delta t}[qf_{\text{up}} + (1-q)f_{\text{down}}]$$

(3)

with

$$q = \frac{e^{r\Delta t}s_{\text{now}} - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}. \quad (4)$$

The argument in *Notes on Options, Part 2* is correct but not very transparent. Here is a more conceptual argument, which shows why the value has to take the special form (3), and gives a simple interpretation for the formula (4).

Our starting point is of course the replicating portfolio, which has $\phi$ units of stock and a cash position worth $\psi$ at the end of the period. The values of $\phi$ and $\psi$ are determined by the two simultaneous linear equations

$$\phi s_{\text{up}} + \psi = f_{\text{up}}$$

$$\phi s_{\text{down}} + \psi = f_{\text{down}}$$

These are two linear equations in two unknowns $\phi$, $\psi$. But let’s be lazy and not solve them explicitly. All we really want to know is the value now of the replicating portfolio, in other words $e^{-r\Delta t}\psi$. This can be found by taking a suitable linear combination of the two equations. What linear combination, exactly? Well, multiplying the first equation by $q_1 e^{r\Delta t}$ and the second by $q_2 e^{r\Delta t}$ then adding gives

$$(q_1 s_{\text{up}} + q_2 s_{\text{down}})\phi + (q_1 + q_2)\psi = e^{-r\Delta t}(q_1 f_{\text{up}} + q_2 f_{\text{down}}). \quad (5)$$

The left hand side has the desired form if

$$q_1 s_{\text{up}} + q_2 s_{\text{down}} = s_{\text{now}} \quad \text{and} \quad q_1 + q_2 = 1.$$  

The latter relation forces $q_2 = 1-q_1$ and the former can be solved to get $q_1 = \frac{e^{r\Delta t}s_{\text{now}} - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}$.  

With these substitutions, (5) becomes the valuation formula (3).

Summary of the preceding: the only way to draw conclusions from a pair of linear equations is to take a linear combination. So the value of the option can only be a weighted combination of $f_{\text{up}}$ and $f_{\text{down}}$. We determined the right weighted combination by asking that its left hand side be the value (now) of the replicating portfolio.

But let’s be even lazier. Suppose you accept, in advance, that the single-period option valuation formula must have the form

$$f_{\text{now}} = e^{-r\Delta t}[qf_{\text{up}} + (1-q)f_{\text{down}}]$$

(6)

for some value of $q$. Is there an easy way to find $q$? There sure is! Consider the special option whose payoff is the same as the value of the stock: $f_{\text{up}} = s_{\text{up}}$ and $f_{\text{down}} = s_{\text{down}}$. Its replicating portfolio clearly consists of one unit of stock and no cash ($\phi = 1$, $\psi = 0$). So its value is $s_{\text{now}}$. Thus if (6) holds for any value of $q$, we must have

$$s_{\text{now}} = e^{-r\Delta t}[qs_{\text{up}} + (1-q)s_{\text{down}}].$$

Solving for $q$ gives the familiar formula (4).
**Some special cases of the multiperiod option pricing formula.** Let’s check the multiperiod pricing formula for consistency with things we already know, and gain some intuition in the process.

*What if the contingent claim pays the stock price itself?* This is the case \( f(s_T) = s_T \). It is replicated by the portfolio consisting of one unit of stock (no bond, no trading). So the present value should be \( s_0 \), the price of the stock now. We used this fact just above, to explain the formula for \( q \) in the single-period binomial setting. Let’s check it now in the multiperiod setting, by verifying that we get the correct value \( (s_0) \) by “working backward through the tree.” It’s enough to show that if \( f(s) = s \) for every possible price \( s \) at a given time then the same relation holds at the time just before. But each time we work backward one timestep we use pricing formula for a single-period binomial tree. So the property \( f(s) = s \) is preserved when we work backward in the tree.

There is of course an equivalent calculation involving risk-neutral expectation. The formula for \( q \) in a multiplicative tree gives

\[
qu + (1 - q)d = e^{r\delta t}
\]

and taking the \( N \)th power gives

\[
\sum_{j=0}^{N} \binom{N}{j} (1 - q)^{j} w^{j} d^{N-j} = e^{rN\delta t} = e^{rT}.
\]

Multiplying both sides by \( s_0 \) gives

\[
e^{-rT} E_{RN}[s_T] = s_0
\]

as desired.

*What if the contingent claim is a forward contract with strike price \( K \)?* Under our standing constant-interest-rate hypothesis we know the present value should be \( s_0 - e^{-rT} K \) if the maturity is \( T = N\delta t \). Let’s verify that any binomial tree gives the same result. The payoff is \( f(s_T) = s_T - K \). Our formula

\[
e^{-rT} E_{RN}[f(s_T)]
\]

is linear in the payoff. Also \( E_{RN}[K] = K \), i.e. the total risk-neutral probability is 1; this can be seen from the fact that \( (q + [1 - q])^N = 1 \). Thus our formula for the value of a forward is

\[
e^{-rT} E_{RN}[s_T - K] = e^{-rT} E_{RN}[s_T] - e^{-rT} E_{RN}[K] = s_0 - e^{-rT} K
\]

as expected.

*What if the contingent claim is a European call with strike price \( K \gg s_0 \)?* We expect such a call to be worthless, or nearly so. This is captured by the model, since only a few
exceptional paths (involving an exceptional number of “ups”) will result in a positive payoff.

What if the contingent claim is a European call with strike price $K \ll s_0$? We expect such a call to be worth about the same as a forward with strike price $K$. This too is captured by the model, since only a few exceptional paths (involving an exceptional number of “downs”) will result in a payoff different from that of the forward.

Analogous observations hold for European puts.

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Passage to the continuous limit. We know that continuous-time lognormal dynamics can be mimicked by a multiplicative binomial tree, with $s_{\text{up}} = u s_{\text{now}}$ and $s_{\text{down}} = d s_{\text{now}}$, by taking $u = e^{\sigma \sqrt{\delta t}}$, $d = e^{-\sigma \sqrt{\delta t}}$, and taking the probability of going up to be $p = \frac{1}{2} \left(1 + \frac{\mu}{\sigma} \sqrt{\delta t}\right)$. We’ll review this calculation in a moment.

We also know that the value of an option is $e^{-rT}$ times the “risk-neutral expected payoff,” i.e. the expected payoff obtained using the same tree but a weighted different coin – one with probability $q = (e^{r\delta t} - d)/(u - d)$ for the up state. In the continuous time limit this clearly corresponds to a lognormal process; to figure out which one, we have only to identify the “risk-neutral” drift, i.e. the value $\mu_{\text{RN}}$ such that

$$q \approx \frac{1}{2} \left(1 + \frac{\mu_{\text{RN}}}{\sigma} \sqrt{\delta t}\right).$$

We’ll see presently that $\mu_{\text{RN}} = r - \frac{1}{2} \sigma^2$.

Let’s start by reviewing why $p = \frac{1}{2} \left(1 + \frac{\mu}{\sigma} \sqrt{\delta t}\right)$ corresponds the lognormal dynamics with drift $\mu$ and volatility $\sigma$. For fixed $T$, the lognormal process has $s(T) = s_0 e^Y$ (equivalently, $\log s(T) = \log s_0 + Y$) where $Y$ is Gaussian with mean $\mu T$ and standard deviation $\sigma \sqrt{T}$.

On the other hand, our $N$-period binomial tree with $\delta t = T/N$ and $u$, $d$, $p$ as indicated above has

$$\log s(T) = \log s_0 + X_1 + \cdots + X_N$$

where each $X_i$ is an independent copy of the random variable

$$X_i = \begin{cases} \sigma \sqrt{\delta t} & \text{with probability } p \\ -\sigma \sqrt{\delta t} & \text{with probability } 1 - p. \end{cases}$$

It’s easy to see that

$$E[X_i] = (2p - 1) \sigma \sqrt{\delta t} = \mu \delta t$$

and

$$\text{Var}(X_i) = (1 - (2p - 1)^2) \sigma^2 \delta t = 4p(1 - p) \sigma^2 \delta t.$$

So $X_1 + \cdots + X_N$ is asymptotically Gaussian, with mean

$$E[X_1] + \cdots + E[X_N] = N \mu \delta t = \mu T$$

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(since \(N\delta t = T\)) and variance
\[
\text{Var}(X_1) + \cdots + \text{Var}(X_N) = N \cdot 4p(1-p)\sigma^2\delta t \to \sigma^2 T
\]
(since \(N\delta t = T\) and \(p \to 1/2\) as \(\delta t \to 0\)). Thus we get the anticipated lognormal behavior in the limit.

OK, now what about the option? There’s no need to repeat the argument: it’s exactly the same, with \(p\) replaced by \(q\). We have only to identify the “risk-neutral drift” \(\mu_{RN}\) defined above. Using Taylor expansion, and dropping all terms smaller than \(\delta t\), we have we have
\[
u = e^{\sigma \sqrt{\delta t}} \approx 1 + \sigma \sqrt{\delta t} + \frac{1}{2} \sigma^2 \delta t
\]
\[
d = e^{-\sigma \sqrt{\delta t}} \approx 1 - \sigma \sqrt{\delta t} + \frac{1}{2} \sigma^2 \delta t
\]
\[
e^{\delta t} \approx 1 + r \delta t
\]
from which it follows that
\[
q = \frac{e^{\delta t} - d}{u - d} \approx \frac{(1 + r \delta t) - (1 - \sigma \sqrt{\delta t} + \frac{1}{2} \sigma^2 \delta t)}{(1 + \sigma \sqrt{\delta t} + \frac{1}{2} \sigma^2 \delta t) - (1 - \sigma \sqrt{\delta t} + \frac{1}{2} \sigma^2 \delta t)} = \frac{\sigma \sqrt{\delta t} + (r - \frac{1}{2} \sigma^2) \delta t}{2 \sigma \sqrt{\delta t}}
\]
whence finally
\[
q \approx \frac{1}{2} \left(1 + \frac{r - \frac{1}{2} \sigma^2}{\sigma} \sqrt{\delta t}\right).
\]
In short: the drift associated with the risk-neutral process is \(\mu_{RN} = r - \frac{1}{2} \sigma^2\), as asserted above. Done!

Let’s be sure we don’t miss the punch line. The value of the option is the discount factor \(e^{-rT}\) times the expected value of the payoff using the risk-neutral probability distribution. In the continuous time limit, the value at time 0 of an option with payoff \(f\) is thus
\[
V(f) = e^{-rT} E\left[f(s_0 e^X)\right]
\]
where \(X\) is a Gaussian random variable with mean \((r - \frac{1}{2} \sigma^2)T\) and variance \(\sigma^2 T\). Equivalently:
\[
V(f) = e^{-rT} \int_{-\infty}^{\infty} f(s_0 e^x) \frac{1}{\sigma \sqrt{2\pi T}} \exp \left[-\frac{(x - [r - \sigma^2/2]T)^2}{2\sigma^2 T}\right] dx.
\]
This (when specialized to puts and calls) is the famous Black-Scholes pricing formula.

Notice that the value of the option depends the volatility of the stock (the parameter \(\sigma\)) but not the drift of the stock (the parameter \(\mu\)). This was already clear in the discrete-time setting. It may seem strange. But remember, we are pricing via arbitrage. The replicating strategy takes no risks. So it doesn’t matter whether we think the stock will go up or down, which is (mainly) what \(\mu\) tells us.

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The Black-Scholes formula for a European call or put. The Black-Scholes pricing formula can be evaluated for any payoff $f$ by numerical integration. But for special payoffs – including the put and the call – we can get explicit expressions in terms of the “cumulative normal”

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$  

($N(x)$ is the probability that a Gaussian random variable with mean 0 and variance 1 has value $\leq x$.) The explicit formulas have advantages over numerical integration: besides being easy to evaluate, they permit us to see quite directly how the value and the hedge portfolio depend on strike price, spot price, risk-free rate, and volatility.

Let

$$c[s_0, T; K] = \text{value at time 0 of a European call with strike } K$$
and maturity $T$, if the spot price is $s_0$;

$$p[s_0, T; K] = \text{value at time 0 of a European put with strike } K$$
and maturity $T$, if the spot price is $s_0$.

The explicit formulas are:

$$c[s_0, T; K] = s_0 N(d_1) - Ke^{-rT} N(d_2)$$
$$p[s_0, T; K] = Ke^{-rT} N(-d_2) - s_0 N(-d_1)$$

in which

$$d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \log(s_0/K) + (r + \frac{1}{2} \sigma^2)T \right]$$
$$d_2 = \frac{1}{\sigma \sqrt{T}} \left[ \log(s_0/K) + (r - \frac{1}{2} \sigma^2)T \right] = d_1 - \sigma \sqrt{T}.$$  

To derive these formulas we use the following result. (The homework problem asking you to calculate $E[e^X]$ for $X$ Gaussian was a special case.)

**Lemma**: Suppose $X$ is Gaussian with mean $\mu$ and variance $\sigma^2$. Then for any real numbers $a$ and $k$,

$$E \left[ e^{ax} \text{ restricted to } X \geq k \right] = e^{a\mu + \frac{1}{2}a^2\sigma^2} N(d)$$

with $d = (-k + \mu + a\sigma^2)/\sigma$.

**Proof**: The left hand side is defined by

$$E \left[ e^{ax} \text{ restricted to } X \geq k \right] = \frac{1}{\sigma \sqrt{2\pi}} \int_{k}^{\infty} e^{ax} \exp \left[ \frac{-(x-\mu)^2}{2\sigma^2} \right] dx.$$  

Complete the square:

$$ax - \frac{(x-\mu)^2}{2\sigma^2} = a\mu + \frac{1}{2}a^2\sigma^2 - \frac{(x-(\mu+az\sigma))^2}{2\sigma^2}.$$
Thus
\[
E \left[ e^{aX} \mid X \geq k \right] = e^{a\mu + \frac{1}{2}a^2\sigma^2} \cdot \frac{1}{\sigma \sqrt{2\pi}} \int_k^\infty \exp \left[ \frac{-(x - (\mu + a\sigma^2))^2}{2\sigma^2} \right] \, dx.
\]

If we set \( u = \left[ x - (\mu + a\sigma^2) \right] / \sigma \) and \( \kappa = \left[ k - (\mu + a\sigma^2) \right] / \sigma \) this becomes
\[
e^{a\mu + \frac{1}{2}a^2\sigma^2} \cdot \frac{1}{\sqrt{2\pi}} \int_\kappa^\infty e^{-u^2/2} \, du = e^{a\mu + \frac{1}{2}a^2\sigma^2} \left[ 1 - N(\kappa) \right] = e^{a\mu + \frac{1}{2}a^2\sigma^2} N(d)
\]
where \( d = -\kappa = (-k + \mu + a\sigma^2) / \sigma \).

We apply this to the European call. Our task is to evaluate
\[
e^{-rT} \int_{-\infty}^\infty (s_0e^x - K) + \frac{1}{\sigma \sqrt{2\pi T}} \exp \left[ \frac{-(x - [r - \sigma^2/2T])^2}{2\sigma^2 T} \right] \, dx.
\]
The integrand is nonzero when \( s_0e^x > K \), i.e., when \( x > \log(K/s_0) \). Applying the Lemma with \( a = 1 \) and \( k = \log(K/s_0) \) we get
\[
e^{-rT} \int_k^\infty s_0e^x \frac{1}{\sigma \sqrt{2\pi T}} \exp \left[ \frac{-(x - [r - \sigma^2/2T])^2}{2\sigma^2 T} \right] \, dx = s_0N(d_1);
\]
applying the Lemma again with \( a = 0 \) we get
\[
e^{-rT} \int_k^\infty K \frac{1}{\sigma \sqrt{2\pi T}} \exp \left[ \frac{-(x - [r - \sigma^2/2T])^2}{2\sigma^2 T} \right] \, dx = Ke^{-rT}N(d_2);
\]
combining these results gives the formula for \( c[s_0, T; K] \).

The formula for the value of a European put can be obtained similarly. Or – easier – we can derive it from the formula for a call, using put-call parity:
\[
p[s_0, T; K] = c[s_0, T; K] + Ke^{-rT} - s_0 = Ke^{-rT} \left[ 1 - N(d_2) \right] - s_0 \left[ 1 - N(d_1) \right] = Ke^{-rT}N(-d_2) - s_0N(-d_1).
\]

For options with maturity \( T \) and strike price \( K \), the value at any time \( t \) is naturally \( c[s_t, T - t; K] \) for a call, \( p[s_t, T - t; K] \) for a put.
Hedging. We know how to hedge in the discrete-time, multiperiod binomial tree setting: the payoff is replicated by a portfolio consisting of \( \Delta = \Delta(0, s_0) \) units of stock and a (long or short) bond, chosen to have the same value as the derivative claim. (We used to write \( \phi \) for the amount of stock in the replicating portfolio. Here we change to the more standard notation \( \Delta \) rather than \( \phi \).) At time \( \delta t \) the stock price changes to \( s_{\delta t} \) and the value of the hedge portfolio changes by \( \Delta(s_{\delta t} - s_0) \). The new value of the hedge portfolio is also the new value of the option, so

\[
\Delta(0, s_0) = \frac{\text{change in value of option from time 0 to } \delta t}{\text{change in value of stock from time 0 to } \delta t}.
\]

The replication strategy requires a self-financing trade at every time step, adjusting the amount of stock in the portfolio to match the new value of \( \Delta \).

In the real world prices are not confined to a binomial tree, and there are no well-defined time steps. We cannot trade continuously. So while we can pass to the continuous time limit for the value of the option, we must still trade at discrete times in our attempts to replicate it. Suppose, for simplicity, we trade at equally spaced times with interval \( \delta t \). What to use for the initial hedge ratio \( \Delta \)? Not being clairvoyant we don’t know the value of the stock at time \( \delta t \), so we can’t use the formula given above. Instead we should use its continuous-time limit:

\[
\Delta(0, s_0) = \frac{\partial}{\partial s_0} \text{(value of option)}.
\]

There’s a subtle point here: if the stock price changes continuously in time, but we only rebalance at discretely chosen times \( j\delta t \), then we cannot expect to replicate the option perfectly using self-financing trades. Put differently: if we maintain the principle that the value of the hedge portfolio is equal to that of the option at each time \( j\delta t \), then our trades will no longer be self-financing. This point can be addressed – but it requires stochastic differential equations, a topic beyond the scope of the present course.

For the European put and call we can easily get formulas for \( \Delta \) by differentiating our expressions for \( c \) and \( p \): at time \( T \) from maturity the hedge ratio should be

\[
\Delta = \frac{\partial}{\partial s_0} c[s_0, T; K] = N(d_1)
\]

for the call, and

\[
\Delta = \frac{\partial}{\partial s_0} p[s_0, T; K] = -N(-d_1)
\]

for the put. The “hard way” to see this is an application of chain rule: for example, in the case of the call,

\[
\frac{\partial}{\partial s_0} c = N(d_1) + s_0 N'(d_1) \frac{\partial d_1}{\partial s} - K e^{-rT} N'(d_2) \frac{\partial d_2}{\partial s}.
\]

But \( d_2 = d_1 - \sigma \sqrt{T} \), so \( \partial d_1/\partial s = \partial d_2/\partial s \); also \( N'(x) = \frac{1}{\sqrt{2\pi}} \exp[-x^2/2] \). It follows with some calculation that

\[
s_0 N'(d_1) \frac{\partial d_1}{\partial s} - K e^{-rT} N'(d_2) \frac{\partial d_2}{\partial s} = 0,
\]
so finally \( \partial c / \partial s_0 = N(d_1) \) as asserted. There is however an easier way: differentiate the original formula expressing the value as a discounted risk-neutral expectation. Passing the derivative under the integral, for a call with strike \( K \):

\[
\Delta = \frac{\partial}{\partial s_0} e^{-rT} \int_{-\infty}^{\infty} (s_0 e^x - K)_+ \frac{1}{\sigma \sqrt{2\pi T}} \exp \left[ \frac{-(x - [r - \sigma^2/2]T)^2}{2\sigma^2 T} \right] dx
\]

\[
= e^{-rT} \int_{-\infty}^{\infty} \frac{\partial(s_0 e^x - K)_+}{\partial s_0} \frac{1}{\sigma \sqrt{2\pi T}} \exp \left[ \frac{-(x - [r - \sigma^2/2]T)^2}{2\sigma^2 T} \right] dx
\]

\[
= e^{-rT} \int_{-\infty}^{\infty} e^x \frac{1}{\sigma \sqrt{2\pi T}} \exp \left[ \frac{-(x - [r - \sigma^2/2]T)^2}{2\sigma^2 T} \right] dx
\]

\[
= N(d_1).
\]