Luenberger discusses maximization of expected utility, but he doesn’t do many examples. Here are two that shed light on the value and limitations of diversification. For further reading on diversification and expected utility, see Chapter 3 (“Time diversification”) of Mark Kritzman’s book *Puzzles of Finance*.

**An example involving simultaneous diversification.** You’re offered the opportunity to flip a coin, receiving 2 dollars if it comes up heads, but owing 1 dollar if it comes up tails. (You need not put up any money to play.) The odds are in your favor, but there is a significant chance of ending up in the red. Should you play? It is tempting to think the answer is yes, at least if you’re allowed to flip many coins rather than just one (this has the effect of diversification – your chance of ending up in the red overall is very much reduced). But if you use optimization of expected utility as the method for making the decision, then the expected result isn’t relevant – it’s the expected utility of the resulting wealth that matters. Therefore the answer depends on your choice of utility. This makes sense: flipping \( N \) coins, there is still a chance of winding up in the red (even a small chance of owing \( N \) dollars!). Your willingness to accept such a risk should enter into the decision-making process. The choice of utility introduces information of this type.

A quantitative assessment is easy if we choose a utility of the form \( \phi(x) = -\exp(-ax) \) with \( a > 0 \). Let

\[
X_i = \begin{cases} 
2 & \text{if the } i\text{th flip comes out heads} \\
-1 & \text{if the } i\text{th flip comes out tails}
\end{cases}
\]

and let \( X = X_1 + \cdots + X_N \) be the wealth produced by \( N \) flips. The coin-flips are independent and identically distributed, so

\[
E\left[-e^{-aX}\right] = -\left(E[e^{-aX_1}]\right)^N = -\theta^N
\]

where \( \theta = E[e^{-aX_1}] \). Flipping is attractive if this greater than the expected utility of not flipping, which is \(-e^{-ax}\) with \( x = 0 \), in other words \(-1\). So flipping is attractive if \( \theta < 1 \). Is it? The answer depends on the choice of \( a \). In fact,

\[
\theta = \frac{1}{2}e^{-2a} + \frac{1}{2}e^a
\]

and Taylor expansion near \( a = 0 \) shows this is less than 1 for \( a > 0 \) sufficiently small, but it is obviously larger than 1 once \( a \) gets big enough.

If \( \theta < 1 \), then flipping coins is attractive. So you should be willing to pay something to do it. How much? Within the framework of expected utility, the value to you is given by the *certainty equivalent*. So you should be willing to pay \( D_N \) dollars to flip \( N \) coins, where \( D_N \) is defined by

\[
-e^{-aD_N} = E\left[-e^{-aX}\right] = -\theta^N,
\]

in other words \( D_N = N|\log \theta|/a \).
An example involving time diversification. Now consider an investment whose return, in any time period, is

\[ R = \begin{cases} 
  2 & \text{with probability } 1/2 \\
  z & \text{with probability } 1/2 
\end{cases} \]

where \( 0 < z < 1 \) is a fixed number. Can it happen that the expected return is greater than 1 but the expected utility of return is less than 1? (If so then investment appears attractive, in terms of the expected return; but actually it is not attractive, on a multiperiod basis, for an investor who makes his decisions based on expected utility.)

The answer is yes, it can happen, for certain values of \( z \). This is almost obvious, since \( E[\phi(R)] < \phi(E[R]) \) when \( \phi \) is (strictly) concave.

For which \( z \) does it happen? That depends on your utility. If it is \( \phi(x) = \log x \) then \( E[\log R] = \log \sqrt{2z} \), so the certainty equivalent return is \( \sqrt{2z} \). (This is what we’ve been calling \( R_{LT} \) – it gives, at least approximately, the long-term rate of growth.) The mean return on the other hand is \( (z + 2)/2 \). The investment has mean return greater than 1 but certainty equivalent return less than 1 if

\[ \sqrt{2z} < 1 < \frac{z + 2}{2}, \]

in other words if \( 0 < z < 1/2 \).

Suppose on the other hand your utility of wealth is \( \phi(x) = \frac{1}{\gamma} x^\gamma \). In this case we must assume distinct times are independent (we didn’t need this assumption for the logarithmic utility – why not?). Then the utility of wealth resulting from \( N \) time-diversification over \( N \) successive periods starting with balance \( D \) is

\[ \frac{1}{\gamma} E[R_N^\gamma \cdots R_1^\gamma D^\gamma] = \frac{1}{\gamma} (E[R_1^\gamma])^N D^\gamma \]

and the certainty-equivalent return is \( (E[R_1^\gamma])^{1/\gamma} \). To make the example explicit, let’s take \( \gamma = -1 \). Then the certainty-equivalent return is \( \frac{4z}{z+2} \) while the mean return is (as before) \( (z + 2)/2 \). The mean return is greater than 1 but the certainty-equivalent return less than 1 if

\[ \frac{4z}{z+2} < 1 < \frac{z + 2}{2}, \]

in other words if \( 0 < z < 2/3 \).

We expect the power law (with negative \( \gamma \)) to be more risk-averse than the logarithmic utility. And indeed it is in this example, since the inequality \( \frac{4z}{z+2} \leq \sqrt{2z} \) (easily verified by elementary algebra) gives

certainty-equiv return using \( \gamma = -1 \) < certainty-equiv return using the log utility.