A two-period binomial example

Let's use a two-period binomial tree to price an option. (People really do use trees to price options; for an accurate answer one should use many periods – perhaps 30 or 40, implemented numerically. But the two-period case is instructive because we can work it out by hand and understand why everything works.)

**Question:** Consider a put option, with strike price $K = 45$ dollars and maturity $T = 1$ year. Suppose the underlying has volatility $\sigma = 20$ percent per annum, and the current spot price is $s_0 = 20$ dollars. Assume the risk-free rate $r$ is 6 percent per annum.

(a) Find the stock prices at the nodes of the binomial tree, using $u = e^{\sigma \sqrt{\Delta t}} = e^{20\sqrt{.5}}$ and $d = e^{-\sigma \sqrt{\Delta t}} = e^{-20\sqrt{.5}}$.

(b) Find the parameter $q$, the “risk-neutral probability” of the up state. Note that $e^{r \Delta t} = e^{.06\cdot.5}$.

(c) Determine the price of the option at each node, by working backward through the tree.

(d) Describe the “trading strategy”, which replicates the option. In other words, specify how many units of stock and how much debt you should hold at each node after “rebalancing.”

(e) Suppose the stock goes up at the initial period. Verify by direct calculation that the trading strategy works.

**Answer:**

(a) With a calculator we find $u = 1.1519$, $d = 0.8681$, and $e^{r \Delta t} = 1.0304$. The stock price tree has price $s_0 = 20$ at time 0, prices $s_0 u = 23.038$ and $s_0 d = 17.362$ at time $\Delta t = 1/2$ year, and $s_0 u^2 = 26.537$, $s_0 ud = 20$, $s_0 d^2 = 15.072$ at time $2 \Delta t = 1$ year.

(b) We find $q = (1.0304 - .8681)/(1.1519 - .8681) = .5719$. Note that $1 - q = .4281$.

(c) At maturity, the option has value $(45 - 26.537)_+ = 18.463$ if the stock price is $s_0 u^2$; value $(45 - 20)_+ = 25$ if the stock price is $s_0 ud$; and $(45 - 15.072)_+ = 29.928$ if the stock price is $s_0 d^2$. Working backward: at the first period the option has value

$$(1.0304)^{-1}[18.463q + 25(1 - q)] = 20.634$$

if the stock price is $s_0 u$, and value

$$(1.0304)^{-1}[25q + 29.928(1 - q)] = 26.310$$

if the stock price is $s_0 d$. At the initial time its value is

$$(1.0304)^{-1}[20.634q + 26.310(1 - q)] = 22.383$$
(d) the replication strategy is to hold $\phi = (f_{up} - f_{down})/(s_{up} - s_{down})$ shares of stock as you go forward. In this example $\phi = -1$ at each node.

(e) The replicating portfolio is always short 1 share of stock (no trading needed). Initially: it contains $-1$ unit stock and 42.383 cash = 22.383 value. At the first time period, cash has earned interest so the portfolio has become $-1$ unit stock and 43.671 cash. Suppose, to fix ideas, the period 1 stock price is 23.038. Then the portfolio value is $43.671 - 23.038 = 20.633$. It matches the value of the option, as expected.

Remark: Actually, since on this tree the final-time prices are all less than $K$, the option is indistinguishable from a short forward contract. We know a forward is equivalent to a portfolio of stock + cash (with no trading needed, and no market model needed). This explains why $\phi = -1$ in this example and why no trading was necessary.