Basic facts about means, variances, and independence

If you have no prior background in probability, the explanations in Luenberger may be somewhat terse. You’ll find a slower introduction to this material in almost any introductory-level probability text. Here are two Dover paperbacks that have the advantage of being (i) at the right level (very basic) and (ii) inexpensive (10 to 15 dollars):

(a) S. Goldberg, Probability: An Introduction, Dover, 1987

(b) J. Freund, Introduction to Probability, Dover, 1993

Luenberger isn’t very good about collecting in a single place the basic facts about means, variances, covariances, independence, etc. Here are the main facts we’ll be using:

(a) \( E[X + Y] = E[X] + E[Y] \). More generally, for \( N \) random variables \( X_1, \ldots, X_N \) with weights \( w_1, \ldots, w_N \), \( E[w_1X_1 + \ldots + w_NX_N] = w_1E[X_1] + \ldots + w_NE[X_N] \). We sometimes write \( \bar{X} \) for \( E[X] \).

(b) \( \text{Var}(X) = E[(X - \bar{X})^2] = E[X^2] - (E[X])^2 \). We sometimes write \( \sigma_X = \sqrt{\text{Var}(X)} \) for the standard deviation of \( X \).

(c) \( \text{Var}(X + Y) = \text{Var}(X) + 2\text{Cov}(X, Y) + \text{Var}(Y) \) with \( \text{Cov}(X, Y) = E[(X - \bar{X})(Y - \bar{Y})] = E[XY] - E[X]E[Y] \). We sometimes write \( \sigma_{XY} \) for \( \text{Cov}(X, Y) \). Notice that order doesn’t matter: \( \text{Cov}(X, Y) = \text{Cov}(Y, X) \).

(d) Generalization of (c): for \( N \) random variables \( X_1, \ldots, X_N \) with weights \( w_1, \ldots, w_N \), \( \text{Var} \left( \sum_{i=1}^{N} w_iX_i \right) = \sum_{i=1}^{N} w_i^2\text{Var}(X_i) + 2\sum_{i<j} w_iw_j\text{Cov}(X_i, X_j) \).

(e) The covariance satisfies \( |\text{Cov}(X, Y)| \leq \sigma_X \sigma_Y \). We sometimes work with the correlation coefficient \( \rho = \text{Cov}(X, Y) / \sigma_X \sigma_Y \); evidently \( |\rho| \leq 1 \).

(f) Two random variables \( X, Y \) are independent if knowing the value of \( X \) gives no information about the value of \( Y \). This corresponds to the probabilities being multiplicative: if \( X \) takes value \( X_\alpha \) with probability \( p_\alpha \) and \( Y \) takes value \( Y_\beta \) with probability \( q_\beta \), then \( X \) and \( Y \) are independent \( \iff \) the probability that \( X \) takes value \( X_\alpha \) and \( Y \) takes value \( Y_\beta \) is \( p_\alpha q_\beta \). If \( X \) and \( Y \) are independent then they are in particular uncorrelated: \( \text{Cov}(X, Y) = 0 \).