These notes discuss why "risk neutral discounted expected payoff" can be used to price (even path-dependent) options.

To keep things concrete I'll focus on our usual classes of (simple) asset price models: lognormal with constant interest rate r, or else
\[ dS = \mu S dt + \sigma S dw \] (still Markovian!). However, the discus extends far beyond that (even to
\[ dS = \mu S dt + \sigma S dw + \Gamma S dt \] where \( \mu, \sigma, \Gamma \) are just measurable, and even to stochastic interest rates).

Besides its intrinsic interest, we need this to explain the "Martingale approach" to portfolio optimization (next week's topic).

Places to read about this:
* the book by Baxter + Rennie
* The book by Karatzas + Shreve

Big picture: we hope this for understand only the "Heston-Kac" leg of the following triangle:
BS pde — replicating portfolio

Feynman Rec
Martingale repn
discounted
risk-neutral
expectation

First, relatively easy goal: explain the "Ito" leg. This applies only to options whose payoff has the form \( k(S(T)) \) (no path dependence), since the Black-Scholes PDE is exact to that setup.

Claim: if \( V(S,t) \) solves BS pde with \( V = 0 \) at \( t = T \), then starting with initial capital \( V(S_0,0) \) at \( t = 0 \), option's payoff can be replicated by a self-financing trading strategy. [So (by absence of arbitrage) option's value at \( t = 0 \) must be \( V(S_0,0) \).]

Proof of claim: consider trading strategy:

- at \( t = 0 \) hold \( \phi_t = \frac{2V}{\partial S} (S_t, t) \) units of stock
- and \( \psi_t = (V(S_t, t) - \phi_t S_t) / B_t \) units of bond

where (since we take the interest rate to be constant)

\( B_t = e^{rt} \) is the value of a risk-free bond at \( t = 0 \).
Clearly its total value is \( V(S_t, t) \). To show it is self-financing we must show that

\[
\frac{dV}{dt} = \sigma^2 S_t \frac{dS}{S_t} + \frac{1}{2} \sigma^2 DB
\]

change in value

profit or loss on stock + bond holdings

LHS is \( \frac{dV}{dt} = V_{dB} + \frac{1}{2} \frac{dV}{ds} dS dS \) by Ito.

RHS is \( \frac{dV}{ds} (V - \frac{dV}{ds} S) r \) since dB = rS dt.

The fact that these are equal is precisely the BS pde.

Of course we know that BS pde is arose to Feynman-Kac applied to \( dS = rS dt + \sigma S dW \).
So we recover this way that option prices are arose to discounted expected payoffs into a
suitable stock process (the "risk-neutral process") different from the subjective one.

But this applies only to path-independent options.
I'd like a result that applies also to path-
derpendent options (ie any \( \mathcal{F} \)-measurable payoff).
Key fact: (Girsanov's Theorem) changing the drift of an SDE amounts to changing the measure we use on path space (and the Radon-Nikodym derivative can be made explicit). In fact, if

$$dS = \mu S \, dt + \sigma S \, dW$$

under the original measure, call it $P$,

Then there's a different measure, call it $Q$, at

$$dS = \rho S \, dt + \sigma S \, d\tilde{W}$$

where $\tilde{W}$ is a $Q$-Brownian motion

(no: $d\tilde{W} = \frac{\mu - \rho}{\sigma} \, dt + dW$) and for every $F_t$-measurable random variable $X$ we have (at time $0$)

$$E_Q[X] = E_P[H_t X]$$

with

$$H_t = \exp \left( -\int_0^t \frac{\mu - \rho}{\sigma} \, dW - \frac{1}{2} \int_0^t \left( \frac{\mu - \rho}{\sigma} \right)^2 \, ds \right)$$

Essence of (1): under the measure $Q$, $S_t/B_t$ is a martingale. Easy consequence of (1) via Itô, since $d(S/B) = \sigma (e^{-rt} S) = (dS - \rho Sdt) e^{-rt} = \sigma (S/B) d\tilde{W}$

Equivalent state:

$$S_t/B_t = E_Q [S_t/B_t \mid F_t]$$
\[ \mathbb{M}_t(S_t/B_t) = \mathbb{E}_p \left[ \mathbb{M}_t S_t/B_t \mid \mathcal{F}_t \right] \]

Easy to see also from form of \( \mathbb{M}_t \) (note this is why \( \mathbb{M}_t \) has its form): need to show \( \mathbb{M}_t(S_t/B_t) \) is a martingale w.r.t. \( \mathbb{P} \)

\[ \mathbb{M}_t = e^{-z_t} \quad dz_t = \frac{z_t}{\sigma} dw + \frac{1}{2} \left( \frac{z_t}{\sigma} \right)^2 dt \]

\[
\Rightarrow \quad d\mathbb{M} = -e^{-z_t} dz_t + \frac{1}{2} e^{-z_t} \left( \frac{z_t}{\sigma} \right)^2 dt \cdot dw \\
= - \mathbb{M} \left( \frac{z_t}{\sigma} \right) dw + \frac{1}{2} \mathbb{M} \left( \frac{z_t}{\sigma} \right)^2 dt \\
= - \mathbb{M} \left( \frac{z_t}{\sigma} \right) dw \\
\]

\[ d(S_t/B_t) = (\mu-r)(S_t/B_t) dt + \sigma(S_t/B_t) dw \]

\[ d(H, S_t/B_t) = H d(S_t/B_t) + \frac{S_t}{B_t} dH + dH d(S_t/B_t) \]

\[
= H \left( \frac{S_t}{B_t} \right) \left[ (\mu-r) dt + \sigma dw \right] \\
- H \left( \frac{S_t}{B_t} \right) \left[ \frac{z_t}{\sigma} \right] \cdot dw \\
- H \left( \frac{z_t}{\sigma} \right) . \sigma \left( \frac{S_t}{B_t} \right) dt \\
= \text{stuff} dw \quad \text{since the "} dt \text{" terms cancel} \]

Why is this useful? We can use it to show (by an arg similar to that given above)
for European options) that the value $V_t$ at time $t$ of a path-dependent option worth $X$ (an $F_t$ measurable random variable) at time $T$ satisfies

$$(**): \quad V_t/B_t = E_Q \left[ X/B_T | F_t \right]$$

in which RHS is conditional expectation w.r.t. into account at time $t$.

To demonstrate (**), we must show existence of a replicating portfolio that's self-financing, whose value at time $t$ is the $V_t$ given by (**).

Key tool: "martingale rep'n thru". Recall that $\sigma_t dW_t$ is a $Q$ martingale. Rep'n thru says every $Q$ martingale has this form. So (using $d(S/B) = \sigma_t (S/B) dW_t$)

$$ (**) \Rightarrow d(V_t/B_t) = \sigma_t d(S_t/B_t)$$

for any ($F_t$-measurable) $\sigma_t$.

Using this, we identify the trading strategy of the replicating portfolio: it holds $q_t$ units of stock + $(V_t - q_t S_t)/B_t$ units of bond at time $t$. 
Value of proposed portfolio in certainty $V_t$. Need to show it is self-financing. In fact,

$$V = \frac{S}{B} \Rightarrow dV = d\left(\frac{S}{B}\right) + \frac{1}{B} dB$$

(I'm using Itô's form $d(xy) = x\,dy + y\,dx + dx\,dy$ and fact that $dx\,dB = 0$ since $dB = \rho dB_t$ has no "dw" term.)

Also,

$$S = \frac{S}{B} \Rightarrow dS = d\left(\frac{S}{B}\right) + \frac{S}{B} dB$$

So, writing $\psi_t = \frac{V_t - \phi_t S_t}{B_t}$, we have

$$\phi_t dS + \gamma_t dB = \psi_t B_t d\left(\frac{S}{B}\right) + \frac{S}{B} dB$$

$$(\frac{V_t}{B} - \phi_t S_t) dB$$

So the choice of $\gamma$ s.t. $\gamma_t d(\frac{S}{B}) = d\left(\frac{V}{B}\right)$ satisfies

$$\phi_t dS + \gamma_t dB = dV$$

Thus: changes in portfolio's value are entirely attributable to market gains/losses + bond interest, as desired.
Rule: since we know how to turn $Q$-expectations into $P$-expectations (using Girsanov), we can also write down the value of an option using the $P$-expectation.

Some examples, to bring this down to earth:

1. stock with constant dividend yield at rate $q$. The tradable in this case is the stock with dividends reinvested.

Claim: if stock price process (under $P$) is

$$dS = \mu S dt + \sigma S dW$$

The RN process (under $Q$) is

$$(*) 
\begin{align*}
    dS &= (r-q) S dt + \sigma S d\tilde{W} \\
\end{align*}$$

The reason is that if you start at $t=0$ with 1 share then your holdings at $t=\tilde{W}$ are $S e^{\tilde{W}}$, and its value is $S e^{\tilde{W}} - q S$. Eqn $(*)$ can be corrected that

$$S e^{\tilde{W}} / B_t = S e^{q t - \frac{1}{2} \sigma^2 t}$$

is a $Q$-martingale.
Suppose US dollar risk-free rate is \( r \)

British pound risk-free rate is \( q \)

exchange rate (dollars/pound) is log-normal

\[ dC = C(t) \sigma \, dt + \sigma C(t) \, dW \]

To dollar investors, a pound looks like "stock with constant yield"; so from eq. 9.2, the dollar-investor's risk-neutral process is \( Q \), where

\[ dC = (r - q) C(t) \, dt + \sigma C(t) \, dW \]

a Q-Brownian motion.

What about the pound investor. His exchange rate is \( 1/C \). By Itô, under the \( P \) measure,

\[ d(1/C) = -C^{-2} \, dC + C^{-3} \, dC \, dC \]

\[ = (-\mu + \sigma^2) \frac{1}{C} \, dt - \sigma \frac{1}{C} \, dW \]

What is the pound investor's risk-neutral measure? Certainly not \( Q \) ! By analogy to what we did for the dollar investor, pound investor's RN measure \( \bar{Q} \) is at

\[ d(1/C) = (q - r) (1/C) \, dt - \sigma (1/C) \, d\bar{W} \]
where $\bar{w}$ is a $\mathcal{Q}$ Brownian motion. Evidently

$$(-\mu + \sigma^2) dt - \sigma dw = (\eta - \phi) dt - \phi \, d\bar{w}.$$ 

whence

$$d\bar{w} = \left(\frac{\mu + \sigma^2 - \phi}{\phi}\right) dt + dw$$

whereas for the dollar investor

$$\mu dt + \sigma \, dw = (\eta - \phi) \, dt + \phi \, d\bar{w}$$

$$\Rightarrow d\bar{w} = \left(\frac{\mu + \sigma^2 - \phi}{\phi}\right) dt + dw$$

Is this strange? Well, no. The "RN measure" is simply the one assuring that (value of tradable $B_t)$ is a martingale. Different investors in this case see different tradables, so they have different RN measures.