Problem 3 corrected 4/8: parts (b) and (c) follow the lead of problem 2, not problem 1.

Except for the first, these problems concern deterministic optimal control (Section 4 material). Some are a bit laborious (though not necessarily difficult). Moreover some students are still completing HW3. Therefore I have fixed the due date as 4/14 (almost 3 weeks).

(1) Consider the standard finite difference scheme
\[
\frac{u_{m}^{n+1} - u_{m}^{n}}{\Delta t} = \frac{u_{m}^{n+1} - 2u_{m}^{n} + u_{m}^{n-1}}{(\Delta x)^2}
\] (1)
for solving \(u_t - u_{xx} = 0\) on an interval. (Here \(u_{m}^{n}\) is the numerical solution at time \(m\Delta t\) and spatial location \(n\Delta x\).) The stability restriction \(\Delta t < \frac{1}{2}(\Delta x)^2\) leaves a lot of freedom in the choice of \(\Delta x\) and \(\Delta t\). Show that \(\Delta t = \frac{1}{6}(\Delta x)^2\) is special, in the sense that the numerical scheme (1) has errors of order \((\Delta x)^4\) rather than \((\Delta x)^2\). In other words: when \(u\) is the exact solution of the PDE, the left and right sides of (1) differ by a term of order \((\Delta x)^4\). [Comment: the argument sketched in Section 3 shows that if \(u\) solves the PDE and \(v\) solves the finite difference scheme then \(|u - v|\) is of order \((\Delta x)^2\) in general, but it is smaller – of order \((\Delta x)^4\) – when \(\Delta t = \frac{1}{6}(\Delta x)^2\), provided that \(u\) is sufficiently smooth.]

(2) Consider the finite-horizon utility maximization problem with discount rate \(\rho\). The dynamical law is thus
\[
dy/ds = f(y(s), \alpha(s)), \quad y(t) = x,
\]
and the optimal utility discounted to time 0 is
\[
u(x, t) = \max_{\alpha \in A} \left\{ \int_{t}^{T} e^{-\rho s} h(y(s), \alpha(s)) \, ds + e^{-\rho T} g(y(T)) \right\}.
\]
It is often more convenient to consider, instead of \(u\), the optimal utility discounted to time \(t\); this is
\[
v(x, t) = e^{\rho t} u(x, t) = \max_{\alpha \in A} \left\{ \int_{t}^{T} e^{-\rho(s-t)} h(y(s), \alpha(s)) \, ds + e^{-\rho(T-t)} g(y(T)) \right\}.
\]
(a) Show (by a heuristic argument similar to those in the Section 4 notes) that \(v\) satisfies
\[
v_t - \rho v + H(x, \nabla v) = 0
\]
with Hamiltonian
\[
H(x, p) = \max_{\alpha \in A} \{ f(x, a) \cdot p + h(x, a) \}.
\]
and final-time data
\[ v(x, T) = g(x). \]
(Notice that the PDE for \( v \) is autonomous, i.e. there is no explicit dependence on time.)

(b) Now consider the analogous infinite-horizon problem, with the same equation of state, and value function
\[
\bar{v}(x, t) = \max_{\alpha \in A} \int_t^\infty e^{-\rho(s-t)} h(y(s), \alpha(s)) \, ds.
\]
Show (by an elementary comparison argument) that \( \bar{v} \) is independent of \( t \), i.e. \( \bar{v} = \bar{v}(x) \) is a function of \( x \) alone. Conclude using part (a) that if \( \bar{v} \) is finite, it solves the stationary PDE
\[
-\rho \bar{v} + H(x, \nabla \bar{v}) = 0.
\]

(3) Recall Example 1 of the Section 4 notes: the state equation is \( \frac{dy}{ds} = ry - \alpha \) with \( y(t) = x \), and the value function is
\[
u(x, t) = \max_{\alpha \geq 0} \int_t^\tau e^{-\rho s} h(\alpha(s)) \, ds
\]
with \( h(a) = a^\gamma \) for some \( 0 < \gamma < 1 \), and
\[
\tau = \left\{ \begin{array}{ll}
\text{first time when } y = 0 & \text{if this occurs before time } T \\
T & \text{otherwise}.
\end{array} \right.
\]
(a) We obtained a formula for \( u(x, t) \) in the Section 4 notes, however our formula doesn’t make sense when \( \rho - r\gamma = 0 \). Find the correct formula in that case.
(b) Let’s examine the infinite-horizon-limit \( T \to \infty \). Following the lead of Problem 2 let us concentrate on \( v(x, t) = e^{\rho t} u(x, t) = \) optimal utility discounted to time \( t \). Show that
\[
\bar{v}(x) = \lim_{T \to \infty} v(x, t) = \left\{ \begin{array}{ll}
G_\infty x^\gamma & \text{if } \rho - r\gamma > 0 \\
\infty & \text{if } \rho - r\gamma \leq 0
\end{array} \right.
\]
with \( G_\infty = [(1 - \gamma)/(\rho - r\gamma)]^{1-\gamma} \).
(c) Use the stationary PDE of Problem 2(b) (specialized to this example) to obtain the same result.
(d) What is the optimal consumption strategy, for the infinite-horizon version of this problem?

(4) Consider the analogue of Example 1 with the power-law utility replaced by the logarithm: \( h(a) = \ln a \). To avoid confusion let us write \( u_\gamma \) for the value function obtained in the notes using \( h(a) = a^\gamma \), and \( u_{\log} \) for the value function obtained using \( h(a) = \ln a \). Recall that \( u_\gamma(x, t) = g_\gamma(t) x^\gamma \) with
\[
g_\gamma(t) = e^{-\rho t} \left[ \frac{1 - \gamma}{\rho - r\gamma} \left( 1 - e^{-\frac{(\rho - r\gamma)(T-t)}{1-\gamma}} \right) \right]^{1-\gamma}.
\]
(a) Show, by a direct comparison argument, that

\[ u_{\log}(\lambda x, t) = u_{\log}(x, t) + \frac{1}{\rho} e^{-\rho t} (1 - e^{-\rho(T-t)}) \ln \lambda \]

for any \( \lambda > 0 \). Use this to conclude that

\[ u_{\log}(x, t) = g_0(t) \ln x + g_1(t) \]

where \( g_0(t) = \frac{1}{\rho} e^{-\rho t} (1 - e^{-\rho(T-t)}) \) and \( g_1 \) is an as-yet unspecified function of \( t \) alone.

(b) Pursue the following scheme for finding \( g_1 \): Consider the utility

\[ h(a) = \frac{1}{\gamma} (a^{\gamma} - 1), \]

Express its value function \( u_h \) in terms of \( u_\gamma \). Now take the limit \( \gamma \to 0 \). Show this gives a result of the expected form, with

\[ g_0(t) = g_\gamma(t)|_{\gamma=0} \]

and

\[ g_1(t) = \frac{dg_\gamma(t)}{d\gamma}|_{\gamma=0}. \]

(This leads to an explicit formula for \( g_1 \) but it’s messy; I’m not asking you to write it down.)

(c) Indicate how \( g_0 \) and \( g_1 \) could alternatively have been found by solving appropriate PDE’s. (Hint: find the HJB equation associated with \( h(a) = \ln a \), and show that the ansatz \( u_{\log} = g_0(t) \ln x + g_1(t) \) leads to differential equations that determine \( g_0 \) and \( g_1 \).)

(5) Our Example 1 considers an investor who receives interest (at constant rate \( r \)) but no wages. Let’s consider what happens if the investor also receives wages at constant rate \( w \). The equation of state becomes

\[ \frac{dy}{ds} = ry + w - \alpha \quad \text{with} \quad y(t) = x, \]

and the value function is

\[ u(x, t) = \max_{\alpha \geq 0} \int_t^T e^{-\rho s} h(\alpha(s)) \, ds \]

with \( h(a) = a^\gamma \) for some \( 0 < \gamma < 1 \). Since the investor earns wages, we now permit \( y(s) < 0 \), however we insist that the final-time wealth be nonnegative (\( y(T) \geq 0 \)).

(a) Which pairs \( (x, t) \) are acceptable? The strategy that maximizes \( y(T) \) is clearly to consume nothing (\( \alpha(s) = 0 \) for all \( t < s < T \)). Show this results in \( y(T) \geq 0 \) exactly if

\[ x + \phi(t)w \geq 0 \]

where

\[ \phi(t) = \frac{1}{r} \left( 1 - e^{-r(T-t)} \right). \]

Notice for future reference that \( \phi \) solves \( \phi' - r\phi + 1 = 0 \) with \( \phi(T) = 0 \).
(b) Find the HJB equation that $u(x, t)$ should satisfy in its natural domain $\{(x, t) : x + \phi(t)w \geq 0\}$. Specify the boundary conditions when $t = T$ and where $x + \phi w = 0$.

(c) Substitute into this HJB equation the ansatz

$$v(x, t) = e^{-\rho t}G(t)(x + \phi(t)w)^\gamma.$$  

Show $v$ is a solution when $G$ solves the familiar equation

$$G_t + (r\gamma - \rho)G + (1 - \gamma)G^{\gamma/(\gamma-1)} = 0$$  

(the same equation we solved in Example 1). Deduce a formula for $v$.

(d) In view of (a), a more careful definition of the value function for this control problem is

$$u(x, t) = \max_{\alpha \geq 0} \int_t^T e^{-\rho s} h(\alpha(s)) \, ds$$

where

$$\tau = \begin{cases} \text{first time when } y(s) + \phi(s)w = 0 \text{ if this occurs before time } T \\ T \text{ otherwise.} \end{cases}$$

Use a verification argument to prove that the function $v$ obtained in (c) is indeed the value function $u$ defined this way.

(6) This problem is a special case of the "linear-quadratic regulator" widely used in engineering applications. The state is $y(s) \in \mathbb{R}^n$, and the control is $\alpha(s) \in \mathbb{R}^n$. There is no pointwise restriction on the values of $\alpha(s)$. The evolution law is

$$\frac{dy}{ds} = Ay + \alpha, \quad y(t) = x,$$

for some constant matrix $A$, and the goal is to minimize

$$\int_t^T |y(s)|^2 + |\alpha(s)|^2 \, ds + |y(T)|^2.$$

(In words: we prefer $y = 0$ along the trajectory and at the final time, but we also prefer not to use too much control.)

(a) What is the associated Hamilton-Jacobi-Bellman equation? Explain why we should expect the relation $\alpha(s) = -\frac{1}{2} \nabla u(y(s))$ to hold along optimal trajectories.

(b) Since the problem is quadratic, it’s natural to guess that the value function $u(x, t)$ takes the form

$$u(x, t) = \langle K(t)x, x \rangle$$

for some symmetric $n \times n$ matrix-valued function $K(t)$. Show that this $u$ solves the Hamilton-Jacobi-Bellman equation exactly if

$$\frac{dK}{dt} = K^2 - I - (K^T A + A^T K) \text{ for } t < T, \quad K(T) = I$$

where $I$ is the $n \times n$ identity matrix. (Hint: two quadratic forms agree exactly if the associated symmetric matrices agree.)

(c) Show by a suitable verification argument that this $u$ is indeed the value function of the control problem.