1. We showed, in the Section 2 notes, that the solution of

\[ w_t = w_{xx} \quad \text{for } t > 0 \text{ and } x > 0, \] with \( w = 0 \) at \( t = 0 \) and \( w = \phi \) at \( x = 0 \)

is

\[
 w(x,t) = \int_0^t \frac{\partial G}{\partial y}(x,0,t-s)\phi(s) \, ds
\]

(1)

where \( G(x,y,s) \) is the probability that a random walker, starting at \( x \) at time 0, reaches \( y \) at time \( s \) without first hitting the barrier at 0. (Here the random walker solves \( dy = \sqrt{2} dw \), i.e. it executes the scaled Brownian whose backward Kolmogorov equation is \( u_t + u_{xx} = 0 \).) Let’s give an alternative demonstration of this fact, following the line of reasoning at the end of the Section 1 notes.

(a) Express, in terms of \( G \), the probability that the random walker (starting at \( x \) at time 0) hits the barrier before time \( t \). Differentiate in \( t \) to get the probability that it hits the barrier at time \( t \). (This is known as the first passage time density).

(b) Use the forward Kolmogorov equation and integration by parts to show that the first passage time density is \( \frac{\partial G}{\partial y}(x,0,t) \).

(c) Deduce the formula (1).

2. As noted in HW2 problem 5, questions about Brownian motion with drift can often be answered using the Cameron-Martin-Girsanov theorem. But we can also study this process directly. Let’s do so now, for the process \( dz = \mu dt + dw \) with an absorbing barrier at \( z = 0 \).

(a) Suppose the process starts at \( z_0 > 0 \) at time 0. Let \( G(z_0,z,t) \) be the probability that the random walker is at position \( z \) at time \( t \) (and has not yet hit the barrier). Show that

\[
 G(z_0,z,t) = \frac{1}{\sqrt{2\pi t}}e^{-[z-z_0-\mu t]^2/2t} - \frac{1}{\sqrt{2\pi t}}e^{-2\mu z_0}e^{-[z-z_0-\mu t]^2/2t}.
\]

(Hint: just check that this \( G \) solves the relevant forward Kolmogorov equation, with the appropriate boundary and initial conditions.)

(b) Show that the first passage time density is

\[
 \frac{1}{2} \frac{\partial G}{\partial z}(z_0,0,t) = \frac{z_0}{t\sqrt{2\pi t}}e^{-|z_0+\mu t|^2/2t}.
\]

3. Consider the linear heat equation \( u_t - u_{xx} = 0 \) on the interval \( 0 < x < 1 \), with boundary condition \( u = 0 \) at \( x = 0,1 \) and initial condition \( u = 1 \).

(a) Interpret \( u \) as the value of a suitable double-barrier option.
(b) Express \( u(t, x) \) as a Fourier sine series, as explained in Section 3.

(c) At time \( t = 1/100 \), how many terms of the series are required to give \( u(t, x) \) within one percent accuracy?

4. Consider the SDE \( dy = f(y)dt + g(y)dw \). Let \( G(x, y, t) \) be the fundamental solution of the forward Kolmogorov PDE, i.e. the probability that a walker starting at \( x \) at time 0 is at \( y \) at time \( t \). Show that if the infinitesimal generator is self-adjoint, i.e.

\[
-(fu)_x + \frac{1}{2}(g^2 u)_{xx} = fu_x + \frac{1}{2}g^2 u_{xx},
\]

then the fundamental solution is symmetric, i.e. \( G(x, y, t) = G(y, x, t) \).

5. Consider the stochastic differential equation \( dy = f(y, s)ds + g(y, s)dw \), and the associated backward and forward Kolmogorov equations

\[
 u_t + f(x, t)u_x + \frac{1}{2}g^2(x, t)u_{xx} = 0 \quad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T
\]

and

\[
 \rho_s + (f(z, s)\rho)_z - \frac{1}{2}(g^2(z, s)\rho)_{zz} = 0 \quad \text{for } s > 0, \text{ with } \rho(z) = \rho_0(z) \text{ at } s = 0.
\]

Recall that \( u(x, t) \) is the expected value (starting from \( x \) at time \( t \)) of payoff \( \Phi(y(T)) \), whereas \( \rho(z, s) \) is the probability distribution of the diffusing state \( y(s) \) (if the initial distribution is \( \rho_0 \)).

(a) The solution of the backward equation has the following property: if \( m = \min_z \Phi(z) \) and \( M = \max_z \Phi(z) \) then \( m \leq u(x, t) \leq M \) for all \( t < T \). Give two distinct justifications: one using the maximum principle for the PDE, the other using the probabilistic interpretation.

(b) The solution of the forward equation does not in general have the same property; in particular, \( \max_z \rho(z, s) \) can be larger than the maximum of \( \rho_0 \). Explain why not, by considering the example \( dy = -ydts \). (Intuition: \( y(s) \) moves toward the origin; in fact, \( y(s) = e^{-s}y_0 \). Viewing \( y(s) \) as the position of a moving particle, we see that particles tend to collect at the origin no matter where they start. So \( \rho(z, s) \) should be increasingly concentrated at \( z = 0 \).) Show that the solution in this case is \( \rho(z, s) = e^{s}\rho_0(e^{s}z) \). This counterexample has \( g = 0 \); can you also give a counterexample using \( dy = -ydts + \epsilon dw \)?

6. The solution of the forward Kolmogorov equation is a probability density, so we expect it to be nonnegative (assuming the initial condition \( \rho_0(z) \) is everywhere nonnegative). In light of Problem 2b it’s natural to worry whether the PDE has this property. Let’s show that it does.

(a) Consider the initial-boundary-value problem

\[
w_t = a(x, t)w_{xx} + b(x, t)w_x + c(x, t)w
\]

with \( x \) in the interval \((0, 1)\) and \( 0 < t < T \). We assume as usual that \( a(x, t) > 0 \). Suppose furthermore that \( c < 0 \) for all \( x \) and \( t \). Show that if \( 0 \leq w \leq M \) at the
initial time and the spatial boundary then $0 \leq w \leq M$ for all $x$ and $t$. (Hint: a positive maximum cannot be achieved in the interior or at the final boundary. Neither can a negative minimum.)

(b) Now consider the same PDE but with $\max_{x,t} c(x,t)$ positive. Suppose the initial and boundary data are nonnegative. Show that the solution $w$ is nonnegative for all $x$ and $t$. (Hint: apply part (a) not to $w$ but rather to $\bar{w} = e^{-Ct}w$ with a suitable choice of $C$.)

(c) Consider the solution of the forward Kolmogorov equation in the interval, with $\rho = 0$ at the boundary. (It represents the probability of arriving at $z$ at time $s$ without hitting the boundary first.) Show using part (b) that $\rho(z,s) \geq 0$ for all $s$ and $z$.

[Comment: statements analogous to (a)-(c) are valid for the initial-value problem as well, when we solve for all $x \in \mathbb{R}$ rather than for $x$ in a bounded domain. The justification takes a little extra work however, and it requires some hypothesis on the growth of the solution at $\infty$.]

7. Consider the solution of

$$u_t + au_{xx} = 0 \quad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T$$

where $a$ is a positive constant. Recall that in the stochastic interpretation, $a$ is $\frac{1}{2}g^2$ where $g$ represents volatility. Let’s use the maximum principle to understand qualitatively how the solution depends on volatility.

(a) Show that if $\Phi_{xx} \geq 0$ for all $x$ then $u_{xx} \geq 0$ for all $x$ and $t$. (Hint: differentiate the PDE.)

(b) Suppose $\bar{u}$ solves the analogous equation with $a$ replaced by $\bar{a} > a$, using the same final-time data $\Phi$. We continue to assume that $\Phi_{xx} \geq 0$. Show that $\bar{u} \geq u$ for all $x$ and $t$. (Hint: $w = \bar{u} - u$ solves $w_t + \bar{a}w_{xx} = f$ with $f = (a - \bar{a})u_{xx} \leq 0$.)