1) Give a probabilistic interpretation for the solution of each PDE. (You must justify your answer to receive full credit.)

(a) \( u_t + f(x)u_x + \frac{1}{2}g^2(x)u_{xx} = 0 \) for \( t < T \) and \( x \in \mathbb{R} \), with final-time condition \( u(x, T) = \phi(x) \).

(b) \( f(x)u_x + \frac{1}{2}g^2(x)u_{xx} = -1 \) on the interval \( a < x < b \), with \( u = 0 \) at the boundary points \( x = a, b \).

2) This problem concerns the explicit solution formulas for the linear heat equation in a half-space and a bounded interval.

(a) The solution of
\[
 u_t = u_{xx} \quad \text{for} \quad t > 0 \quad \text{and} \quad x > 1, \quad \text{with} \quad u = (x-1)^3 \quad \text{at} \quad t = 0 \quad \text{and} \quad u = 0 \quad \text{at} \quad x = 1
\]
can be expressed as
\[
 u(x, t) = \frac{1}{\sqrt{4\pi t}} \int e^{-|x-y|^2/4t} \phi(y) \, dy.
\]
What is \( \phi(y) \)?

(b) The solution of
\[
 u_t = u_{xx} \quad \text{for} \quad t > 0 \quad \text{and} \quad 0 < x < 1, \quad \text{with} \quad u = g(x) \quad \text{at} \quad t = 0 \quad \text{and} \quad u = 0 \quad \text{at} \quad x = 0, 1
\]
can be expressed as
\[
 u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin(n\pi x).
\]
Find \( a_n(t) \) in terms of \( g \).

3) This problem concerns the arrival time at the boundary, for a random walker solving \( dy = f dt + gw \) on the interval \([a, b]\).

(a) Let \( G(x, y, t) \) be the probability, starting from \( x \) at time 0, of being at \( y \) at time \( t \) without having yet hit the boundary. What version of the forward Kolmogorov equation does \( G \) solve?

(b) Express, as an integral involving \( G_t \), the “first passage time density to the boundary,” i.e. the probability that the process, starting from \( a < x < b \), first hits the boundary at time \( t \).

(c) Using your answers to (a) and (b) and some further manipulation, show that
\[
 \text{first passage time density to the boundary} = -\frac{1}{2} \frac{\partial}{\partial y} (g^2 G(x, y, t)) \bigg|_{y=b} + \frac{1}{2} \frac{\partial}{\partial y} (g^2 G(x, y, t)) \bigg|_{y=a}.
\]

4) Consider the following version of the Merton asset allocation problem:

- There is a risk-free asset, whose price satisfies \( dp_1 = rp_1 ds \).

- There are two risky assets, whose prices \( p_2 \) and \( p_3 \) satisfy \( dp_i = \mu_i p_i ds + \sigma_i p_i dw_i \) for \( i = 2, 3 \). We assume for simplicity that \( w_2 \) and \( w_3 \) are independent Brownian motions.

- Your controls are \( \alpha_i(s) = \) the fraction of your wealth invested in asset \( i \) at time \( s \), \( i = 1, 2, 3 \); note that \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \).

- There is no consumption, and your goal is to optimize your expected utility of wealth at a predetermined time \( T \). Your utility function is \( h \).
Answer the following:

(a) What stochastic differential equation describes the evolution of your total wealth?
(b) Define an appropriate value function \( u(x,t) \).
(c) Specify the Hamilton-Jacobi-Bellman equation and final-time condition \( u \) should satisfy.
(d) How does the value function determine the optimal asset allocations \( \alpha_i \)?

5) In pricing a perpetual American put, we considered an underlying satisfying \( dy = \mu y ds + \sigma y dw \) and the goal was to evaluate \( \max_{\tau} E_{y(0) = x} \left[ e^{-\tau r} (K - y(\tau))_+ \right] \). Show that if \( v \) is a differentiable function with \( v \geq (K - x)_+ \) and \(-rv + \mu vx + \frac{1}{2}\sigma^2 x^2 v_{xx} \leq 0 \) for all \( x \) then \( v \) gives an upper bound:

\[
E_{y(0) = x} \left[ e^{-\tau r} (K - y(\tau))_+ \right] \leq v(x)
\]

for any bounded, nonanticipating stopping time \( \tau \).

6) This is a variant of the Bertsimas-Kogan-Lo least-squares-replication problem considered in Section 7. It differs from the version in the notes in two ways: (i) the underlying has stochastic volatility; and (ii) the goal is not least-square replication but rather maximizing the utility of final-time wealth.

The underlying is a stock which can be traded at discrete times \( i\Delta t \). Its price \( P_i \) and volatility \( \sigma_i \) at the \( i \)th time satisfy

\[
\sigma_{i+1} = \sigma_i + f(\sigma_i)\Delta t + g(\sigma_i)\phi_i \sqrt{\Delta t} \\
P_{i+1} = P_i + \sigma_i P_i \psi_i \sqrt{\Delta t}
\]

where \( f \) and \( g \) are specified functions and \( \psi_i, \phi_i \) are independent standard Gaussians (with mean 0 and variance 1).

You have sold an option on this stock with payoff \( F(P_N) \), receiving cash \( V_0 \) in payment. Your goal is to invest this cash wisely, trading in a self-financing way, to maximize the expected utility of your final-time wealth \( E[h(V_N - F(P_N))] \). Here \( h \) is your utility.

(a) Set this up as a discrete-time optimal control problem. What are the state variables? What is the control? Define an appropriate value function (call it \( J_i \)) at time \( i\Delta t \). Be sure to specify the arguments of \( J_i \), i.e. the variables it depends on.
(b) What is the value of \( J_N \)?
(c) Give a recursion relation that specifies \( J_i \) in terms of \( J_{i+1} \) for \( i < N \).

7) Consider scaled Brownian motion with jumps: \( dy = \sigma dw + J dN \), starting at \( y(0) = x \). Assume the jump occurrences are Poisson with rate \( \lambda \), and the jumps have mean 0 and variance \( \delta^2 \).

(a) Find \( E[y^2(T)] \). (Hint: for a Poisson process with rate \( \lambda \), the expected number of arrivals by time \( T \) is \( \lambda T \).)
(b) What backward Kolmogorov equation does part (a) solve?