1) Our geometric Example 2 gave $|\nabla u| = 1$ in $D$ (with $u = 0$ at $\partial D$) as the HJB equation associated with starting at a point $x$ in some domain $D$, traveling with speed at most 1, and arriving at $\partial D$ as quickly as possible. Let’s consider what becomes of this problem when we introduce a little noise. The state equation becomes

$$dy = \alpha(s)ds + \epsilon dw, \quad y(0) = x,$$

where $\alpha(s)$ is a (non-anticipating) control satisfying $|\alpha(s)| \leq 1$, $y$ takes values in $\mathbb{R}^n$, and each component of $w$ is an independent Brownian motion. Let $\tau_{x,\alpha}$ denote the arrival time:

$$\tau_{x,\alpha} = \text{time when } y(s) \text{ first hits } \partial D,$$

which is of course random. The goal is now to minimize the expected arrival time at $\partial D$, so the value function is

$$u(x) = \min_{|\alpha(s)| \leq 1} \mathbb{E}_{y(0)=x} \{ \tau_{x,\alpha} \}.$$

(a) Show, using an argument similar to that in the Section 3 notes, that $u$ solves the PDE

$$1 - |\nabla u| + \frac{1}{2} \epsilon^2 \Delta u = 0 \quad \text{in } D$$

with boundary condition $u = 0$ at $\partial D$.

(b) Your answer to (a) should suggest a specific feedback strategy for determining $\alpha(s)$ in terms of $y(s)$. What is it?

2) Let’s solve the differential equation from the last problem explicitly, for the special case when $D = [-1, 1]$:

$$1 - |u_x| + \frac{1}{2} \epsilon^2 u_{xx} = 0 \quad \text{for } -1 < x < 1$$

$$u = 0 \quad \text{at } x = \pm 1.$$

(a) Assuming that the solution $u$ is unique, show it satisfies $u(x) = u(-x)$. Conclude that $u_x = 0$ and $u_{xx} < 0$ at $x = 0$. Thus $u$ has a maximum at $x = 0$.

(b) Notice that $v = u_x$ solves $1 - |v| + \delta v_x = 0$ with $\delta = \frac{1}{2} \epsilon^2$. Show that

$$v = -1 + e^{-x/\delta} \quad \text{for } 0 < x < 1$$

$$v = +1 - e^{x/\delta} \quad \text{for } -1 < x < 0.$$

Integrate once to find a formula for $u$.

(c) Verify that as $\epsilon \to 0$, this solution approaches $1 - |x|$.
[Comment: the assumption of uniqueness in part (a) is convenient, but it can be avoided. Outline of how to do this: observe that any critical point of $u$ must be a local maximum (since $u_x = 0$ implies $u_{xx} < 0$). Therefore $u$ has just one critical point, say $x_0$, which is a maximum. Get a formula for $u$ by arguing as in (b). Then use the boundary condition to see that $x_0$ had to be 0.]

3) Let’s consider what becomes of Merton’s optimal investment and consumption problem if there are two risky assets: one whose price satisfies $dp_2 = \mu_2 p_2 dt + \sigma_2 p_2 dw_2$ and another whose price satisfies $dp_3 = \mu_3 p_3 dt + \sigma_3 p_3 dw_3$. To keep things simple let’s suppose $w_2$ and $w_3$ are independent Brownian motions. It is natural to assume $\mu_2 > r$ and $\mu_3 > r$ where $r$ is the risk-free rate. (Why?) Let $\alpha_2(s)$ and $\alpha_3(s)$ be the proportions of the investor’s total wealth invested in the risky assets at time $s$, so that $1 - \alpha_2 - \alpha_3$ is the proportion of wealth invested risk-free. Then the investor’s wealth satisfies

$$dy = (1 - \alpha_2 - \alpha_3)yds + \alpha_2 y(\mu_2 ds + \sigma_2 dw_2) + \alpha_3 y(\mu_3 ds + \sigma_3 dw_3).$$

(Be sure you understand this; but you need not explain it on your solution sheet.) Use the power-law utility: the value function is thus

$$u(x, t) = \max_{\alpha_2, \alpha_3} E_{y(t) = x} \left[ \int_t^T e^{-\rho s} \beta(y(s)) ds \right]$$

where $\tau$ is the first time $y(s) = 0$ if this occurs, or $\tau = T$ otherwise.

(a) Derive the HJB equation.

(b) What is the optimal investment policy (the optimal choice of $\alpha_2$ and $\alpha_3$)? What restriction do you need on the parameters to be sure $\alpha_2 > 0, \alpha_3 > 0$, and $\alpha_2 + \alpha_3 < 1$?

(c) Find a formula for $u(x, t)$. [Hint: the nonlinear equation you have to solve is not really different from the one considered in Section 3.]

4) Problem 8 of Homework 2 was a special case of the deterministic “linear quadratic regulator” problem. Here is the analogous stochastic problem. The state is $y(s) \in \mathbb{R}^n$, and the control is $\alpha(s) \in \mathbb{R}^n$. There is no pointwise restriction on the possible value of $\alpha(s)$. The evolution law is

$$dy = (Ay + \alpha)ds + \epsilon dw,$$

where $w$ is a vector-valued Brownian motion (each component is a scalar-valued Brownian motion, and different components are independent). The initial condition is $y(t) = x$, and the goal is to minimize (among nonanticipating controls) the expected cost

$$E_{y(t) = x} \left\{ \int_t^T [||y(s)||^2 + |\alpha(s)|^2] ds + |y(T)|^2 \right\}.$$

The interpretation is similar to the deterministic case: we prefer $y = 0$ for $t < s < T$ and at the final time $T$, but we also prefer not to use too much control. The new element is that the state keeps getting jostled by the noise $\epsilon dw$. 

2
(a) Find the associated HJB equation. Explain why the relation \( \alpha(s) = -\frac{1}{2}\nabla u(y(s)) \) should hold for the optimal control. (Same relation as in the deterministic case!)

(b) Look for a solution of the form

\[ u(x, t) = \langle K(t)x, x \rangle + q(t) \]

where \( K(t) \) is symmetric-matrix-valued and \( q(t) \) is scalar-valued. Show that this \( u \) solves the HJB equation exactly if

\[ \frac{dK}{dt} = K^2 - I - (K^T A + A^T K) \text{ for } t < T, \quad K(T) = I \]

(same as the deterministic case), and

\[ \frac{dq}{dt} = -\epsilon^2 \operatorname{tr} K(t) \text{ for } t < T, \quad q(T) = 0. \]

(c) Show that \( K(t) \) is positive definite. (Hint: its quadratic form is the value function of the deterministic control problem.) Conclude that \( q(t) > 0 \) for \( t < T \).

(d) Show by a verification argument that this \( u \) is indeed the value function of the control problem.

[Comment: in this setting the control law for the stochastic case, \( \alpha(s) = -K(s)y(s) \), is the same as for the deterministic one. However the expected cost is higher due to the term \( q(t) \).]