Math Finance I Notes – Section 7
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Lognormal price dynamics and passage to the continuum limit. This section introduces the lognormal model of stock price dynamics, and explains how it can be approximated using binomial trees. Then we use these binomial trees to price contingent claims. The Black-Scholes analysis is obtained in the limit $\delta t \to 0$. As usual, Baxter–Rennie captures the central ideas concisely (Section 2.4) while Jarrow–Turnbull has a more leisurely treatment (Chapter 4, supplemented by Section 5.5). Hull has a lot of information about the lognormal model scattered through Chapters 10 and 11, including discussions of estimating volatility from historical data (Section 11.3), estimating volatility from market prices of options (Section 11.10), and experimental tests of the lognormal hypothesis (Section 11.11).

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Lognormal stock price dynamics. Our simple model of a risk-free asset has a constant interest rate. A bond worth $\psi_0$ dollars at time 0 is worth $\psi(t) = \psi_0 e^{rt}$ dollars at time $t$. The quantity that’s constant is not the growth rate $\frac{d\psi}{dt}$ but rather the interest rate $r = \frac{1}{\psi} \frac{d\psi}{dt} = \frac{d\log \psi}{dt}$.

Our stock is risky, i.e. its evolution is unknown and appears to be random. We can still describe its dynamics in terms of an equivalent interest rate for each time period. Breaking time up into intervals of length $\delta t$, the equivalent interest rate for $j \delta t < t < (j + 1) \delta t$ is $r_j$ if $s((j + 1)\delta t) = e^{r_j \delta t} s(j\delta t)$, i.e.

$$r_j = \frac{\log s((j + 1)\delta t) - \log s(j\delta t)}{\delta t}.$$  

Standard terminology: $r_j$ is the return of the stock over the relative time interval. [Beware of possible confusion: in our discussion of one-period models the “return” on the risk-free asset was $e^{rT}$; in our present terminology the “return” on the risk-free asset is $r$. This sloppiness is not my fault; it’s standard usage.] Note that to calculate the stock price change over a longer interval you just add the exponents:

$$s(k\delta t) = e^{(r_j \delta t + r_{j+1} \delta t + \ldots + r_{k-1} \delta t)} s(j\delta t), \quad \text{for } j < k.$$  

Since the stock price is random so is each $r_j$. The lognormal model of stock price dynamics specifies their statistics:

- The random variables $r_j \delta t$ are independent, identically distributed, Gaussian random variables with mean $\mu \delta t$ and variance $\sigma^2 \delta t$, for some constants $\mu$ and $\sigma$.

The constant $\mu$ is called the expected return (though actually, the expected return over a time interval of length $\delta t$ is $\mu \delta t$). The constant $\sigma$ is called the “volatility of return,” or more briefly just volatility. These constants are assumed to be the same regardless of the length of the interval $\delta t$. Thus we really mean the following slightly stronger statement:
For any time interval \((t_1, t_2)\), \(\log s(t_2) - \log s(t_1)\) is a Gaussian random variable with mean \(\mu(t_2 - t_1)\) and variance \(\sigma^2(t_2 - t_1)\).

The Gaussian random variables associated with disjoint time intervals are independent.

In particular (for those who know what this means) \(\log s(t)\) executes a Brownian motion with drift. Strictly speaking \(\sigma\) has units of \(1/\sqrt{\text{time}}\), however it is common to call \(\sigma\) the “volatility per year”.

Why should we believe this hypothesis about stock prices? Perhaps it would be more credible to suppose that the daily (or hourly or minute-by-minute) return is determined by a random event (arrival of news, perhaps) which we can model by flipping a coin. The lognormal model is the limit of such dynamics, as the time-frequency of the coin-flips tends to zero. We’ll discuss this in detail presently.


The lognormal hypothesis will lead us to a formula for the present value of a derivative security – but it’s important to remember that the formula is no better than the stock price model it’s based on. The formula doesn’t agree perfectly with what one finds in the marketplace; the main reason is probably that the lognormal model isn’t a perfect model of real stock prices. Much work has been done on improving it – for example by letting the volatility itself be random rather than constant in time.

Lognormal dynamics and the limit of multiperiod binomial trees. We claim that lognormal dynamics can be approximated by dividing time into many intervals, and flipping a coin to determine the return for each interval.

The coin can be fair or biased; to keep things as simple as possible let’s concentrate on the fair case first. To simulate a lognormal process with expected return \(\mu\) and volatility \(\sigma\) the return should be

\[
\begin{align*}
\mu \delta t + \sigma \sqrt{\delta t} & \quad \text{if heads (probability 1/2)} \\
\mu \delta t - \sigma \sqrt{\delta t} & \quad \text{if tails (probability 1/2)}.
\end{align*}
\]

In other words, given \(\delta t\) we wish to consider the recombinant binomial tree with with

\[
s_{\text{up}} = s_{\text{now}} e^{\mu \delta t + \sigma \sqrt{\delta t}}, \quad s_{\text{down}} = s_{\text{now}} e^{\mu \delta t - \sigma \sqrt{\delta t}}
\]

and with each branch assigned (subjective) probability 1/2.
Consider any time $t$. What is the probability distribution of stock prices at time $t$? Let’s assume for simplicity that $t$ is a multiple of $\delta t$, specifically $t = n\delta t$. If in arriving at this time you got heads $j$ times and tails $n - j$ times, then the stock price is

$$s(0) \exp \left[ n\mu\delta t + j\sigma\sqrt{\delta t} - (n - j)\sigma\sqrt{\delta t} \right] = s(0) \exp \left[ \mu t + (2j - n)\sigma\sqrt{t} \right].$$

We should be able to understand the probability distribution (asymptotically as $\delta t \to 0$), since we surely understand the results of flipping a coin many times. Briefly: if you make a histogram of the proportion of heads, it will resemble (as $n \to \infty$) a Gaussian distribution centered at $1/2$. We’ll get the variance straight in a minute. (What we’re really using here is the central limit theorem.)

To proceed more quantitatively it’s helpful to use the notation of probability. Recognizing that $j$ is a random variable, let’s change notation to make it look like one by calling it $X_n$:

$$X_n = \text{number of times you get heads in } n \text{ flips of a fair coin.}$$

Since $X_n$ is the sum of $n$ independent random variables (one for each coin-flip), each taking values 0 and 1 with probability $1/2$, one easily sees that

- Expected value of $X_n = n/2$,
- Variance of $X_n = n/4$.

Thus our histogram, which was the distribution function of $\frac{1}{n}X_n$, tended to a Gaussian with mean $\frac{1}{2}$ and variance $\frac{1}{4n}$. It’s easy to see from this that

$$\frac{2X_n - n}{\sqrt{n}}$$

tends to a Gaussian with mean value 0 and variance 1. Since $\sqrt{\delta t} = \sqrt{t}/\sqrt{n}$ our formula for the final stock price can be expressed as

$$s(t) = s(0) \exp \left[ \mu t + \sigma\sqrt{t} \frac{2X_n - n}{\sqrt{n}} \right].$$

Thus asymptotically, as $\delta t \to 0$ and $n \to \infty$ with $t = n\delta t$ held fixed,

$$s(t) = s(0) \exp \left[ \mu t + \sigma\sqrt{t} Z \right]$$

where $Z$ is a random variable with mean 0 and variance 1. In particular $\log s(t) - \log s(0)$ is a Gaussian random variable with mean $\mu t$ and variance $\sigma^2 t$, as expected.

Our assertion of lognormal dynamics said a little more: that $\log s(t_2) - \log s(t_1)$ was Gaussian with mean $\mu(t_2 - t_1)$ and variance $\sigma^2(t_2 - t_1)$ for all $t_1 < t_2$. The justification is the same as what we did above – it wasn’t really important that we started at 0.

Notice that our calculation used only the mean and variance of $X_n$, since it was based on the Central Limit Theorem. Our particular way of choosing the tree – with $s_{\text{up}} = s_{\text{now}}e^{\mu\delta t + \sigma\sqrt{\delta t}}$, $s_{\text{down}} = s_{\text{now}}e^{\mu\delta t - \sigma\sqrt{\delta t}}$, and with each choice having probability 1/2, was not the only one possible. A more general approach would take $s_{\text{up}} = s_{\text{now}}u$ with probability $p$, $s_{\text{down}} = s_{\text{now}}d$ with probability $1 - p$, and choose the three parameters $u, d, p$ to satisfy two constraints associated with the mean and variance. Evidently one degree of freedom remains. Thus once $p$ is fixed the other parameters are determined.
Implication for pricing options. We attached subjective probabilities (always equal to 1/2) to our binomial tree because we wanted to recognize lognormal dynamics as the limit of a coin-flipping process. Now let us consider one of those binomial trees – for some specific \( \delta t \) near 0 – and use it to price options.

The structure of the tree remains relevant (particularly the factors \( u \) and \( d \) determining \( s_{\text{up}} = us_{\text{now}} \) and \( s_{\text{down}} = ds_{\text{now}} \)). The subjective probabilities (1/2 for every branch) are irrelevant because our pricing is based on arbitrage. But we know a formula for the price of the option with payoff \( f(s(T)) \) at time maturity \( T \):

\[
V(f) = e^{-rT} \cdot E_{\text{RN}}[f(s_T)]
\]

where \( E_{\text{RN}} \) denotes the expected value relative to the risk-neutral probability. And using the risk-neutral probability instead of the subjective probability just means our coin is no longer fair. Instead it is biased, with probability of heads (stock goes up) \( q \) and probability of tails (stock goes down) \( 1 - q \), where

\[
q = \frac{e^{r\delta t} - d}{u - d} = \frac{e^{r\delta t} - e^{\mu \delta t - \sigma \sqrt{\delta t}}}{e^{\mu \delta t + \sigma \sqrt{\delta t}} - e^{\mu \delta t - \sigma \sqrt{\delta t}}}.
\]

One verifies (using the Taylor expansion of \( e^x \) near \( x = 0 \)) that this is close to 1/2 when \( \delta t \) is small, and in fact

\[
q = \frac{1}{2} \left( 1 - \sqrt{\delta t} \frac{\mu - r + \frac{1}{2} \sigma^2}{\sigma} \right) + \text{terms of order } \delta t.
\]

Our task is now clear. All we have to do is find the distribution of final values \( s(T) \) when one uses the \( q \)-biased coin, then take the expected value of \( f(s(T)) \) with respect to this distribution. We can use a lot of what we did above: writing \( X_n \) for the number of heads as before, we still have

\[
s(t) = s(0) \exp \left[ \mu t + \sigma \sqrt{t} \frac{2X_n - n}{\sqrt{n}} \right].
\]

But now \( X_n \) is the sum of \( n \) independent random variables with mean \( q \) and variance \( q(1-q) \), so \( X_n \) has mean \( nq \) and variance \( nq(1-q) \). So

\[
\text{mean of } \frac{2X_n - n}{\sqrt{n}} = (2q - 1)\sqrt{n}
\]

\[
\approx -\sqrt{t} \left( \frac{\mu - r + \frac{1}{2} \sigma^2}{\sigma} \right)
\]

and

\[
\text{variance of } \frac{2X_n - n}{\sqrt{n}} \approx 1.
\]
The central limit theorem tells us the limiting distribution is Gaussian, and the preceding calculation tells us its mean and variance. In summary: as $\delta t \to 0$, when using the biased coin associated with the risk-neutral probabilities,

$$s(t) = s(0) \exp \left[ \mu t + \sigma \sqrt{t} Z' \right]$$

where $Z'$ is a Gaussian random variable with mean $\sqrt{t} \left( \frac{r - \mu - \frac{1}{2} \sigma^2}{\sigma} \right)$ and variance 1. Equivalently, writing $Z' = Z + \sqrt{t} \left( \frac{r - \mu - \frac{1}{2} \sigma^2}{\sigma} \right)$,

$$s(t) = s(0) \exp \left[ (r - \frac{1}{2} \sigma^2) t + \sigma \sqrt{t} Z \right]$$

where $Z$ is Gaussian with mean 0 and variance 1. Notice that the statistical distribution of $s(t)$ depends on $\sigma$ and $r$ but not on $\mu$ (we’ll return to this point soon).

The value of the option is the $e^{-rT}$ times the expected value of the payoff relative to this probability distribution. Using the distribution function of the Gaussian to evaluate the expected value, we get:

$$V(f) = e^{-rT} \int_{-\infty}^{\infty} f(s_0 e^x) \frac{1}{\sigma \sqrt{2\pi T}} \exp \left[ \frac{-(x - [r - \sigma^2 / 2]T)^2}{2\sigma^2 T} \right] dx.$$  

This (when specialized to puts and calls) is the famous Black-Scholes relation. We’ll talk later about evaluating the integral. For now let’s be satisfied with working backward through the binomial tree obtained with a specific (small) value of $\delta t$. Reviewing what we found above: given a lognormal stock process with return $\mu$ and volatility $\sigma$, and given a choice of $\delta t$, the tree should be constructed so that $s_{\text{up}} = u s_{\text{now}}$, $s_{\text{down}} = d s_{\text{now}}$ with

$$u = e^{\mu \delta t + \sigma \sqrt{\delta t}}, \quad d = e^{\mu \delta t - \sigma \sqrt{\delta t}}.$$

These determine the risk-neutral probability $q$ by the formula given above. Working backward through the tree is equivalent to finding the discounted expected value of $f(s(T))$ relative to the risk-neutral probability.

Let us return to the observation, made above, that the statistics of $s(t)$ relative to the risk-neutral probability depend on $\sigma$ (volatility of the stock) and $r$ (risk-free return) but not on $\mu$. It follows that for pricing derivative securities the value of $\mu$ isn’t really needed. More precisely: in the limit $\delta t \to 0$ the lognormal stock models with different $\mu$’s but the same $\sigma$ all assign the same values to options. So we may choose $\mu$ any way we please – there’s no reason to require that it match the actual expected return of the stock under consideration. The two most common choices are

1. choose $\mu$ to be the expected return of the stock nevertheless; or
2. choose $\mu$ so that $\mu - r + \frac{1}{2} \sigma^2 = 0$, i.e. $\mu = r - \frac{1}{2} \sigma^2$. 

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The latter choice has the advantage that it puts \( q \) even closer to 1/2. This is the selection favored by Jarrow–Turnbull and many other authors.

It may seem strange that the value of an option doesn’t depend on \( \mu \). Heuristic argument why this should be so: we are using arbitrage considerations, so it doesn’t matter whether the stock tends to go up or down, which is (mainly) what \( \mu \) tells us. More honest argument why this should be so: suppose we already know (e.g. using part of the argument given above) that the distribution of \( s(t) \) relative to the risk-neutral probability is lognormal with volatility \( \sigma \) and some return \( \tilde{\mu} \). Recall from the beginning of Section 6 that

\[
e^{-rT}E_{RN}[s(T)] = s(0)
\]

because there’s a replicating trading strategy (hold one unit of stock and never trade). I claim that when \( s(T) \) is lognormal with volatility \( \sigma \) and return \( \tilde{\mu} \) we have

\[
E_{RN}[s(T)] = s(0) \exp[\tilde{\mu}T + \sigma^2 T/2]
\]

Comparing these two formulas, we see that the value of \( \tilde{\mu} \) is determined:

\[
\tilde{\mu} = \mu - \sigma^2/2,
\]

in agreement with our direct calculation done earlier.

It remains to justify the preceding claim. Let’s drop the tilde’s: our starting point is the knowledge that \( s(T) = s(0)e^X \) where \( X \) is a Gaussian random variable with mean \( \mu T \) and variance \( \sigma^2 T \). Then the expected value of \( s(T) \) is

\[
s(0)\frac{1}{\sigma\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{x} e^{-\frac{(x-\mu T)^2}{2\sigma^2 T}} dx.
\]

Complete the square:

\[
x - \frac{(x - \mu T)^2}{2\sigma^2 T} = \mu T + \frac{1}{2} \sigma^2 T - \frac{(x - [\mu T + \sigma^2 T])^2}{2\sigma^2 T}.
\]

Therefore the expected value of \( s(T) \) is

\[
s(0)e^{[\mu T + \sigma^2 T/2]} \frac{1}{\sigma\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-\frac{(x-\tilde{\mu}T+\sigma^2 T)^2}{2\sigma^2 T}} dx.
\]

Making the change of variable \( u = \frac{(x-\tilde{\mu}T+\sigma^2 T)}{\sigma\sqrt{T}} \) this becomes

\[
s(0)e^{[\mu T + \sigma^2 T/2]} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du = s(0)e^{[\mu T + \sigma^2 T/2]}
\]

as desired.