The general one-period market model. The binomial and trinomial (single-period) models are special cases of a more general theory, which we now present. Main purposes of this discussion:

- deeper understanding of risk-neutral probabilities;
- more careful treatment of the “principle of no arbitrage”;
- preparation for the Arbitrage Pricing Theory view of CAPM, and more generally for factor models.

This material is standard, but I don’t know a good elementary source. Advanced sources include Chapter 2 of J. Ingersoll, “Theory of financial decision making,” Rowman and Littlefield, 1987; and Chapter 1 of D. Duffie, “Dynamic asset pricing theory”, Princeton University Press, 1996.

In considering one-period models with few assets and many states, we are close to the issue of portfolio analysis: which of the many possible portfolios should an investor hold? We’ll address that issue soon, in the discussion of CAPM. In concentrating on the one-period case we miss an important dynamic aspect of investing – the ability to change asset allocations as time evolves. The final sections of Luenberger’s book give an elementary but informative discussion of how things look a bit different when viewed dynamically.

The general one-period market has

- \( N \) securities, \( i = 1, \ldots, N \)
- \( M \) final states, \( \alpha = 1, \ldots, M \)
- fixed initial values: one unit of security \( i \) is worth \( p_i \) dollars
- state-dependent final values: if the final state is \( \alpha \) then one unit of security \( i \) is worth \( D_{i\alpha} \).

An investor can hold any portfolio \( \theta_i \) units of security \( i \). It has initial value \( \langle p, \theta \rangle = \sum p_i \theta_i \). If the final state is \( \alpha \) then its final value is \( \langle \theta, D_{i\alpha} \rangle = \sum \theta_i D_{i\alpha} \).

Examples:

**Binomial model:** \( p = (e^{-rT}, s_1), \quad D = \begin{pmatrix} 1 & 1 \\ s_2 & s_3 \end{pmatrix} \)

**Trinomial model:** \( p = (e^{-rT}, s_1), \quad D = \begin{pmatrix} 1 & 1 & 1 \\ s_2 & s_3 & s_4 \end{pmatrix} \)
In general, if security 1 is a riskless bond then

\[ p = (e^{-rT}, p_2, \ldots, p_N), \quad D = \begin{pmatrix} 1 & \cdots & 1 \\ D_{21} & \cdots & D_{2M} \\ \vdots & \ddots & \vdots \\ D_{N1} & \cdots & D_{NM} \end{pmatrix} \]

Here’s a careful statement of the **Principle of no arbitrage**:

(a) \( \sum_i \theta_i D_{i\alpha} \geq 0 \) for all \( \alpha \) \( \implies \sum_i \theta_i p_i \geq 0 \)

(b) when the conclusion of (a) holds with \( = \) then the hypothesis must also have \( = \) for every \( \alpha \).

These capture with precision the informal statements that (a) a portfolio with nonnegative payoff has nonnegative value; and (b) a portfolio with nonnegative and sometimes positive payoff has strictly positive value.

The key result relating risk-neutral probabilities to lack of arbitrage is this:

**Theorem**: The economy satisfies (a) iff there exist \( \pi_\alpha \geq 0 \) such that

\[ \sum_\alpha D_{i\alpha} \pi_\alpha = p_i, \quad i = 1, \ldots, N. \]

It satisfies both (a) and (b) if in addition the \( \pi_\alpha \) can be chosen to be all strictly positive.

The theorem is trivial in one direction: assuming the existence of \( \pi_\alpha \) we can easily prove the absence of arbitrage. In fact, for any portfolio \( \theta_i \) we have

\[ \sum_i \theta_i D_{i\alpha} \geq 0 \text{ for all } \alpha \implies \sum_{i,\alpha} \theta_i D_{i\alpha} \pi_\alpha \geq 0 \]

\[ \implies \sum_i \theta_i p_i \geq 0 \]

since \( \pi_\alpha \geq 0 \). If \( \pi_\alpha > 0 \) for each \( \alpha \) then the conclusion can hold with \( = \) only if each hypothesis holds with \( = \) rather than \( \geq \).

We now show that property (a) implies existence of \( \pi_\alpha \geq 0 \). A proof can be given using the min/max version of duality we used for the trinomial model. However I think it’s more convincing to use the following result. Its proper name is Farkas’ Lemma, but I think of it as the Fundamental Lemma of Linear Programming. (See e.g. V. Chvatal, *Linear Programming*, W.H. Freeman 1983, pg. 248, for this and related results).

**Fundamental Lemma of Linear Programming**: If a collection of linear inequalities implies another linear inequality then it does so “trivially,” i.e. the conclusion is a (non-negative) linear combination of the hypotheses.
Now, property (a) says that the collection of linear inequalities \( \sum_i \theta_i D_{i\alpha} \geq 0 \) for \( \alpha = 1, \ldots, M \) implies another linear inequality \( \sum_i \theta_i p_i \geq 0 \). By the Fundamental Lemma of Linear Programming this occurs only if there is a “trivial” proof, i.e. if there exists \( \pi_\alpha \geq 0 \) such that \( \sum \theta_i p_i = \sum_{i, \alpha} \theta_i D_{i\alpha} \pi_\alpha \) for all \( \theta_i \). But that means \( \sum D_{i\alpha} \pi_\alpha = p_i \).

Our final task is to show that if the economy satisfies both (a) and (b) then we can take \( \pi_\alpha > 0 \) for all \( \alpha \). If the \( \pi_\alpha \) already identified are all positive then we’re done. If not, then renumbering states if necessary we may suppose \( \pi_1, \ldots, \pi_{M'} > 0 \) and \( \pi_{M'+1} = \ldots = \pi_M = 0 \).

Let’s concentrate for a moment on index \( M'+1 \). If \( D_{M'+1} = (D_{1M'+1}, \ldots, D_{NM'+1}) \) is a linear combination of \( D_1, \ldots, D_{M'} \) then we can easily modify \( \pi_\alpha \) to make \( \pi_{M'+1} > 0 \). In fact, suppose \( D_{M'+1} = b_1 D_1 + \ldots + b_{M'} D_{M'} \). Then

\[
p_i = \sum_{\alpha=1}^{M'} D_{i\alpha} \pi_\alpha = \epsilon D_{iM'+1} + \sum_{\alpha=1}^{M'} D_{i\alpha} (\pi_\alpha - \epsilon b_\alpha),
\]

so replacing \( \pi = (\pi_1, \ldots, \pi_{M'}, 0, \ldots, 0) \) with \( (\pi_1 - \epsilon b_1, \ldots, \pi_{M'} - \epsilon b_{M'}, \epsilon, 0, \ldots, 0) \) does the trick when \( \epsilon \) is sufficiently small.

Essentially the same argument shows that if any positive combination of \( D_{M'+1}, \ldots, D_M \) lies in the span of \( D_1, \ldots, D_{M'} \) then we can modify \( \pi_\alpha \) to make additional components positive.

Applying the preceding argument finitely many times, we either arrive at a new \( \pi \) with strictly positive components, or we find ourselves in a situation (with a new value of \( M' \)) where no positive combination of \( D_{M'+1}, \ldots, D_M \) lies in the span of \( D_1, \ldots, D_{M'} \). We claim the second alternative cannot happen when the economy has property (b).

This is another application of the Fundamental Lemma of Linear Programming. Our “second alternative” is that

\[
\sum_{\alpha=M'+1}^{M} a_\alpha D_{.\alpha} = \sum_{\alpha=1}^{M'} b_\alpha D_{.\alpha}, \quad a_\alpha \geq 0 \quad \Rightarrow \quad a_\alpha = 0, \alpha = M'+1, \ldots, M.
\]

The “trivial consequences” of the hypotheses are obtained by taking linear combinations. This amounts to taking the inner product with a vector \( \theta \in \mathbb{R}^N \). Thus the trivial consequences of the hypotheses are

\[
\sum_{\alpha=M'+1}^{M} a_\alpha \langle D_{.\alpha}, \theta \rangle = \sum_{\alpha=1}^{M'} b_\alpha \langle D_{.\alpha}, \theta \rangle.
\]
For this (coupled with $a_\alpha \geq 0$) to give a trivial proof that $a_\alpha = 0$ we must have

$$
\langle D_\alpha, \theta \rangle = \sum_i \theta_i D_{i\alpha} = 0 \quad \alpha = 1, \ldots, M'
$$
$$
\langle D_\alpha, \theta \rangle = \sum_i \theta_i D_{i\alpha} > 0 \quad \alpha = M' + 1, \ldots, M.
$$

But then $\theta$ represents a portfolio with no downside, some upside, and value 0 since $\sum_i \theta_i p_i = \sum_i \sum_{\alpha=1}^{M'} \theta_i D_{i\alpha} \pi_\alpha = 0$. This contradicts our assumption that the economy admits no arbitrage.

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Our last application of the Fundamental Lemma of Linear Programming can alternatively be obtained using this geometrically intuitive result from convex analysis:

**Separating hyperplane theorem**: Let $\mathcal{C}$ be a closed convex cone in $\mathbb{R}^N$, and let $\mathcal{L}$ be a linear subspace meeting $\mathcal{C}$ only at the origin. Then there exists a codimension-one hyperplane $\mathcal{H}$ containing $\mathcal{L}$ which meets $\mathcal{C}$ only at the origin.

In our setting the convex cone $\mathcal{C}$ consists of positive linear combinations of $D_{M'+1}, \ldots, D_M$, and the subspace $\mathcal{L}$ is spanned by $D_1, \ldots, D_{M'}$. The associated $\mathcal{H}$ has the form $\{x : \langle \theta, x \rangle = 0\}$, with $\theta$ as above.

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To connect this with risk-neutral probabilities, let us assume that Security 1 is a risk-less bond. Then $p_1 = e^{-rT}$ and the first row of $D_{i\alpha}$ is filled with 1’s. The statement of the theorem becomes: the market permits no arbitrage iff there exist positive $\pi_\alpha$ such that

$$
\pi_1 + \cdots + \pi_M = e^{-rT}
$$
$$
\sum_{\alpha} \pi_\alpha D_{i\alpha} = p_i, \quad i = 2, \ldots, N.
$$

Writing $\hat{\pi}_\alpha = e^{rT} \pi_\alpha$ we see that this is equivalent to the existence of positive $\hat{\pi}_\alpha$ such that

$$
\hat{\pi}_1 + \cdots + \hat{\pi}_M = 1
$$
$$
\sum_{\alpha} \hat{\pi}_\alpha D_{i\alpha} = e^{rT} p_i, \quad i = 2, \ldots, N.
$$

These $\hat{\pi}_\alpha$ are the risk-neutral probabilities.
For the trinomial market, we showed how arbitrage considerations restrict the initial value of any contingent claim $f$. The same max/min argument works in general, for any market in which Security 1 is a riskless bond. The conclusion is

$$\min_{\text{risk-neutral probs } \tilde{\pi}} e^{-rT} \sum_{\alpha} \tilde{\pi}_\alpha f_\alpha \leq V(f) \leq \max_{\text{risk-neutral probs } \tilde{\pi}} e^{-rT} \sum_{\alpha} \tilde{\pi}_\alpha f_\alpha.$$ 

We immediately see that

market completeness $\Leftrightarrow$ arbitrage determines the value of every contingent claim

$\Leftrightarrow$ there is a unique risk-neutral probability.

**Exercises**

1. Consider a forward contract on a non-dividend-paying stock, with strike price $K$ and maturity $T$. Its value at time 0 is $s_0 - Ke^{-rT}$, where $r$ is the risk-free rate (assumed constant) and $s_0$ is the stock price at time 0. We explained this in Section 1, using the standard “cash-and-carry” argument. Explain how that argument can be formalized using a one-period model with two assets and $M$ states.

2. Consider the following one-period market with 3 assets and 4 states:
   - Asset 1 is a riskless bond, paying no interest.
   - Asset 2 is a stock with initial price 1 dollar/share; its possible final prices are $d$ and $u$, with $d < 1 < u$.
   - Asset 3 is another stock with initial price 1 dollar/share and possible final prices $d$ and $u$ (same $d$ and $u$).
   - To keep the arithmetic simple, let’s assume that $u = 1 + \epsilon$ and $d = 1 - \epsilon$ for some $\epsilon > 0$. To avoid confusion, let’s number the states: 1 = both stocks go up; 2 = asset 2 goes up, asset 3 goes down; 3 = asset 2 goes down, asset 3 goes up; 4 = both stocks go down.

   (a) What is the associated cash-flow matrix $D_{i\alpha}$?
   (b) Find all the risk-neutral probabilities.
   (c) Consider the contingent claim with payoff $f = (f_1, f_2, f_3, f_4)$. What are the smallest and largest prices for $f$ permitted by arbitrage considerations? (Let’s call these $V_-(f)$ and $V_+(f)$.)
   (d) Does $f_\alpha \geq 0$ for all $\alpha$ and $V_-(f) = 0$ imply $f = 0$? Explain.
   (e) Which $f$’s are replicatable?

3. It is said that a London betmaker gave the following odds on the 1996 US Presidential election: 6-1 in favor of Clinton, 7-2 against Dole, 50-1 against Perot. Interpret this to mean that the betmaker was willing to take only three types of bets — that Clinton would win, that Dole would win, and that Perot would win — and
- 6 dollars bet on Clinton would return 7 if he won, 0 if not;
- 2 dollars bet on Dole would return 9 if he won, 0 if not;
- 1 dollar bet on Perot would return 51 if he won, 0 if not.

Interpret this as a one-period market with three assets: a 1-dollar bet on Clinton, a 1-dollar bet on Dole, and a 1-dollar bet on Perot. What are the associated risk-neutral probabilities? How much was the betmaker taking of every dollar bet? Explain. (This problem is adapted from Marco Avellaneda’s notes.)