Forwards, puts, calls, and other contingent claims. This section discusses the most basic examples of contingent claims, and explains how considerations of arbitrage determine or restrict their prices. This material is in Chapters 2 and 3 of Jarrow and Turnbull, and Chapters 1, 3 and 7 of Hull. We discuss only European options now, postponing American options till later.

The most basic instruments:

**Forward contract** with maturity T and delivery price K.

- buy a forward $\leftrightarrow$ hold a long forward
- $\leftrightarrow$ holder is obliged to buy the
  underlying asset at price K on date T.

**European call option** with maturity T and strike price K.

- buy a call $\leftrightarrow$ hold a long call
- $\leftrightarrow$ holder is entitled to buy the
  underlying asset at price K on date T.

**European put option** with maturity T and strike price K.

- buy a put $\leftrightarrow$ hold a long put
- $\leftrightarrow$ holder is entitled to sell the
  underlying asset at price K on date T.

These are contingent claims, i.e. their value at maturity is not known in advance. Payoff formulas and diagrams (value at maturity, as a function of $S_T$=value of the underlying) are shown in the Figure.

Figure 1: Payoffs of forward, call, and put options.
Any long position has a corresponding (opposite) short position:

Buyer of a claim has a long position ↔ seller has a short position.

Payoff diagram of short position = negative of payoff diagram of long position.

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Why do people buy and sell contingent claims? Briefly, to speculate or to hedge. Examples of hedges:

- A US airline has a contract to buy a French airplane for a price fixed in FF, payable one year from now. By going long on a forward contract for FF (payable in dollars) it can eliminate its foreign currency risk.

- The holder of a forward contract has unlimited downside risk. Holding a call limits the downside risk (but buying a call with strike K costs more than buying the forward with delivery price K). Holding one long call and one short call costs less, but gives up some of the upside benefit:

\[(S_T - K_1)_+ - (S_T - K_2)_+ \quad K_1 < K_2\]

This is known as a “bull spread”. (See the figure.)

![Figure 2: Payoff of a bull spread.](image)

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Some pricing principles:

- If two portfolios have the same payoff then their present values must be the same.

- If portfolio 1’s payoff is always at least as good as portfolio 2’s, then present value of portfolio 1 ≥ present value of portfolio 2.
We’ll see presently that these principles must hold, because if they didn’t the market would support arbitrage.

**First example: value of a forward contract.** We assume for simplicity:

(a) underlying asset pays no dividend and has no carrying cost (e.g. a non-dividend-paying stock);

(b) time value of money is computed using compound interest rate \( r \), i.e. a guaranteed income of \( D \) dollars time \( T \) in the future is worth \( e^{-rT}D \) dollars now.

The latter hypothesis amounts to introducing one more investment option:

**Bond** worth \( D \) dollars at maturity \( T \)

- buy a bond \( \rightarrow \) hold a long bond
- \( \leftrightarrow \) lend \( e^{-rT}D \) dollars, to be repaid at time \( T \) with interest.

Consider these two portfolios:

**Portfolio 1** – one long forward with maturity \( T \) and delivery price \( K \), payoff \( (S_T - K) \).

**Portfolio 2** – long one unit of stock (present value \( S_0 \), value at maturity \( S_T \)) and short one bond (present value \( -Ke^{-rT} \), value at maturity \( -K \)).

They have the same payoff, so they must have the same present value. Conclusion:

\[
\text{Present value of forward} = S_0 - Ke^{-rT}.
\]

In practice, forward contracts are normally written so that their present value is 0. This fixes the delivery price: \( K = S_0e^{rT} \) where \( S_0 \) is the spot price.

We can see why the “pricing principles” enunciated above must hold. If the market price of a forward were different from the value just computed then there would be an arbitrage opportunity:

- forward is overpriced \( \rightarrow \) sell portfolio 1, buy portfolio 2
  \( \rightarrow \) instant profit at no risk
- forward is underpriced \( \rightarrow \) buy portfolio 1, sell portfolio 2
  \( \rightarrow \) instant profit at no risk.

In either case, market forces (oversupply of sellers or buyers) will lead to price adjustment, restoring the price of a forward to (approximately) its no-arbitrage value.

**Second example: put–call parity.** Define

\[
p(S_0, T, K) = \text{price of European put when spot price is } S_0, \text{ strike price is } K, \text{ maturity is } T
\]

\[
c(S_0, T, K) = \text{price of European call when spot price is } S_0, \text{ strike price is } K, \text{ maturity is } T.
\]
The Black-Scholes model gives formulas for $p$ and $c$ based on a certain model of how the underlying security behaves. But we can see now that $p$ and $c$ are related, without knowing anything about how the underlying security behaves (except that it pays no dividends and has no carrying cost). “Put-call parity” is the relation

$$p[S_0, T, K] = c[S_0, T, K] + Ke^{-rT} - S_0.$$ 

To see this, compare

**Portfolio 1** – one long put; payoff is $(K - S_T)_+.$

**Portfolio 2** – one long call + one short stock + one long bond paying $K$ at maturity; payoff is $(S_T - K)_+ - S_T + K = (K - S_T)_+.$

These portfolios have the same payoff, so they must have the same present value. This justifies the formula.

**Third example:** The prices of European puts and calls satisfy

$$c[S_0, T, K] \geq (S_0 - Ke^{-rT})_+ \quad \text{and} \quad p[S_0, T, K] \geq (Ke^{-rT} - S_0)_+.$$ 

To see the first relation, observe first that $c[S_0, T, K] \geq 0$ by optionality – holding a long call is never worse than holding nothing. Observe next that $c[S_0, T, K] \geq S_0 - Ke^{-rT}$, since holding a long call is never worse than holding the corresponding forward contract. Thus $c[S_0, T, K] \geq \max\{0, S_0 - Ke^{-rT}\}$, which is the desired conclusion. The argument for the second relation is similar.

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Note some hypotheses underlying our discussion:

- no transaction costs; no bid-ask spread
- unlimited possibility of long and short positions; no restriction on borrowing.

These are of course merely approximations to the truth (like any mathematical model). More accurate for large institutions than for individuals.

Note also some features of our discussion: We are simply reaping consequences of the hypothesis of no arbitrage. Conclusions reached this way don’t depend at all on what you think the market will do in the future. Arbitrage methods restrict the prices of (related) instruments. But they don’t tell an individual investor how best to invest his money.

**Exercises.** See Chapters 2 and 3 of Jarrow and Turnbull for a nice selection of exercises on this material.