Derivative Securities – Homework 4 – distributed 10/25/04, due 11/08/04

These problems provide some practice with the Black-Scholes PDE (Problem 1) and stochastic differential equations (Problems 2-5).

1) We considered, in HW3, a derivative whose payoff was \( s^n(T) \) at maturity, where \( s(t) \) has lognormal dynamics with constant volatility \( \sigma \), and the risk-free rate is \( r \) (also constant).

We showed there that the derivative has value

\[
s^n(t) \exp \left( \left( \frac{1}{2} \sigma^2 (n - 1) + r(n - 1) \right) (T - t) \right)
\]

at time \( t \). Let’s give a different derivation of the same result, using the Black-Scholes PDE.

(a) Substitute \( V(s, t) = h(t) s^n \) into the Black-Scholes PDE. What ODE must \( h(t) \) solve? What is the appropriate final-time condition?

(b) Verify that \( h(t) = \exp \left( \left( \frac{1}{2} \sigma^2 (n - 1) + r(n - 1) \right) (T - t) \right) \) solves the ODE you found in (a), with the appropriate final-time condition.

(2) Consider the solution of

\[
\frac{ds}{dt} = r(t)s dt + \sigma(t)s dw, \quad s(0) = s_0.
\]  (1)

where \( r(t) \) and \( \sigma(t) \) are deterministic functions of time.

(a) Show that log \( s(t) \) is a Gaussian random variable, with mean \( \int_0^t [r(s) - \frac{1}{2} \sigma^2(s)] ds \) and variance \( \int_0^t \sigma^2(s) ds \).

(b) Show that \( s(T) = s_0 \exp \left( \left[ r - \frac{1}{2} \sigma^2 T + \sigma \sqrt{T} Z \right] \right) \) where \( Z \) is a standard Gaussian,

\[
\tau = \frac{1}{T} \int_0^T r(s) ds \quad \text{and} \quad \sigma^2 = \frac{1}{T} \int_0^T \sigma^2(s) ds.
\]

[Comment: we’ll show soon that (1) is the “risk-neutral” stock price process when the risk-free rate and volatility are deterministic functions of \( t \). This problem shows that options can be valued in that setting using the standard Black-Scholes formula, with \( r \) replaced by \( \tau \) and \( \sigma \) replaced by \( \sigma \).]

(3) We showed in class using Ito’s formula that if \( s(t) = s(0)e^{\mu t + \sigma w(t)} \) then \( ds = (\mu + \frac{1}{2} \sigma^2) s dt + \sigma s dw \).

(a) Conclude that \( E[s(t)] - E[s(0)] = (\mu + \frac{1}{2} \sigma^2) \int_0^t E[s(\tau)] d\tau \), where \( E \) denotes expected value.

(b) Conclude that \( E[s(t)] = s(0)e^{(\mu + \frac{1}{2} \sigma^2)t} \).
[Comment: taking \( t = 1 \), this gives a new proof of the lemma, stated at the end of the Section 4 notes, that if \( X \) is Gaussian with mean \( \mu \) and standard deviation \( \sigma \) then \( E[e^X] = e^{\mu + \sigma^2/2} \).]

(4) This problem should help you understand Ito’s formula. If \( w \) is Brownian motion, then Ito’s formula tells us that \( z = w^2 \) satisfies the stochastic differential equation \( dz = 2wdw + dt \). Let’s see this directly:

(a) Suppose \( a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b \). Show that \( w^2(t_{i+1}) - w^2(t_i) = 2w(t_i)(w(t_{i+1}) - w(t_i)) + (w(t_{i+1}) - w(t_i))^2 \), whence

\[
w^2(b) - w^2(a) = 2 \sum_{i=0}^{N-1} w(t_i)(w(t_{i+1}) - w(t_i)) + \sum_{i=0}^{N-1} (w(t_{i+1}) - w(t_i))^2
\]

(b) Let’s assume for simplicity that \( t_{i+1} - t_i = (b-a)/N \). Find the mean and variance of \( S = \sum_{i=0}^{N-1} (w(t_{i+1}) - w(t_i))^2 \).

(c) Conclude by taking \( N \to \infty \) that

\[
w^2(b) - w^2(a) = 2 \int_a^b w \, dw + (b-a).
\]

[Comment: we did parts of this calculation in the notes and in class, but because it’s so enlightening I’m asking you to go through it carefully here.]

(5) Here’s a cute application of the Ito calculus. Let

\[
\beta_k(t) = E[w^k(t)]
\]

where \( w(t) \) is Brownian motion (with \( w(0) = 0 \)). Show using Ito’s formula that for \( k = 2, 3, \ldots \),

\[
\beta_k(t) = \frac{1}{2} k(k-1) \int_0^t \beta_{k-2}(s) \, ds.
\]

Deduce that \( E[w^4(t)] = 3t^2 \). What is \( E[w^6(t)] \)?

[Comment: the moments of \( w \) can also be calculated from its distribution function, since \( w(t) \) is Gaussian with mean 0 and variance 1. But the method in this problem is easier, and good practice with Ito’s lemma.)}