This section discusses the Hull-White model. The continuous-time analysis is not much more difficult than Vasicek – everything is still quite explicit. The basic paper is Rev. Fin. Stud. 3, no. 4 (1990) 573-592, downloadable via JSTOR; my treatment is much simpler though because I keep the parameters $a$ and $\sigma$ constant rather than letting them (as well as $\theta$) vary with time.

The real importance of Hull-White is that while it’s rich enough to match any forward curve, it’s also simple enough to be approximated by a (recombining, trinomial) tree. This topic is covered very clearly in Sections 23.11-23.12 of Hull (5th edition), so I won’t cover it separately in these notes.

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**The Hull-White model.** Also sometimes known as “extended Vasicek,” this model assumes that the risk-neutral process for the short rate has the form

$$dr = (\theta(t) - ar) dt + \sigma dw,$$

where $a$ and $\sigma$ are constant but $\theta$ is a function of $t$. (Actually the 1990 paper by Hull and White also considers taking $a = a(t)$ and $\sigma = \sigma(t)$.) We’ll show that

(a) for a given choice of $\theta(t)$, the situation is a lot like Vasicek;

(b) there is a unique choice of $\theta$ that matches the term structure observed in the marketplace at $t = 0$.

**Solving for $r(t)$.** The calculation is entirely parallel to Vasicek: we have

$$d(e^{at}r) = e^{at} dr + ae^{at} r dt = \theta(t)e^{at} dt + e^{at}\sigma dw,$$

so

$$e^{at}r(t) = r(0) + \int_0^t \theta(s)e^{as} ds + \sigma \int_0^t e^{as} dw(s).$$

which simplifies to

$$r(t) = r(0)e^{-at} + \int_0^t \theta(s)e^{-a(t-s)} ds + \sigma \int_0^t e^{-a(t-s)} dw(s).$$

That calculation could have started at any time; thus

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t \theta(\tau)e^{-a(t-\tau)} ds + \sigma \int_s^t e^{-a(t-\tau)} dw(\tau).$$

Notice that $r(t)$ is still Gaussian.
Solving for $P(t, T)$. We use the same PDE method that worked for Vasicek. We know that $P(t, T) = V(t, r(t))$ where $V$ solves the PDE

$$V_t + (\theta(t) - ar)V_r + \frac{1}{2}\sigma^2 V_{rr} - rV = 0$$

with final-time condition $V(T, r) = 1$ for all $r$ at $t = T$. Let’s look for a solution of the form

$$V = A(t, T)e^{-B(t, T)r(t)}.$$  \hspace{1cm} (3)

To satisfy the PDE, $A$ and $B$ should satisfy

$$A_t - \theta(t)AB + \frac{1}{2}\sigma^2 AB^2 = 0 \quad \text{and} \quad B_t - aB + 1 = 0$$

with final-time conditions

$$A(T, T) = 1 \quad \text{and} \quad B(T, T) = 0.$$  \hspace{1cm} (4)

The equation for $B$ doesn’t involve $\theta$, so the solution is the same as for Vasicek:

$$B(t, T) = \frac{1}{a} \left( 1 - e^{-a(T-t)} \right).$$

The equation for $A$ is different only in the fact that $\theta$ is no longer constant; not surprisingly, the $\theta$-dependent part of the solution formula requires doing an integration:

$$A(t, T) = \exp \left[ - \int_t^T \theta(s)B(s, T) \, ds - \frac{\sigma^2}{2a^2}(B(t, T) - T + t) - \frac{\sigma^2}{4a}B(t, T)^2 \right]. \hspace{1cm} (5)$$

**Determining $\theta$ from the term structure at time 0.** Our goal is to demonstrate that following relation between the infinitesimal forward rate and the function $\theta(t)$:

$$\theta(t) = \frac{\partial f}{\partial T}(0, t) + af(0, t) + \frac{\sigma^2}{2a}(1 - e^{-2at}).$$  \hspace{1cm} (6)

When we get to HJM we’ll find a simple proof of this relation. But we can also prove it now, by using the explicit representation of $P(t, T)$ given by (3)-(5). Recall that $f(t, T) = -\partial \log P(t, T)/\partial T$. We have

$$- \log P(0, T) = \int_0^T \theta(s)B(s, T) \, ds + \frac{\sigma^2}{2a^2}(B(0, T) - T) + \frac{\sigma^2}{4a}B(0, T)^2 + B(0, T)r_0.$$  

Differentiating and using that $B(T, T) = 0$ and $\partial_T B - 1 = -aB$, we get

$$f(0, T) = \int_0^T \theta(s)\partial_T B(s, T) \, ds - \frac{\sigma^2}{2a}B(0, T) + \frac{\sigma^2}{2a}B(0, T)\partial_T B(0, T) + \partial_T B(0, T)r_0.$$  

Differentiating again, we get

$$\partial_T f(0, T) = \theta(T) + \int_0^T \theta(s)\partial_{TT} B(s, T) \, ds - \frac{\sigma^2}{2a}\partial_T B(0, T)$$

$$+ \frac{\sigma^2}{2a}[(\partial_T B(0, T))^2 + B(0, T)\partial_{TT} B(0, T)] + \partial_{TT} B(0, T)r_0.$$
Combining these equations, and using the fact that \(a\partial_T B + \partial_T T B = 0\), we get

\[ af(0, T) + \partial_T f(0, T) = \theta(T) - \frac{\sigma^2}{2a}(aB + \partial_T B) + \frac{\sigma^2}{2a}[aB\partial_T B + (\partial_T B)^2 + B\partial_T T B]. \]

Substituting the formula for \(B\) and simplifying, we finally get

\[ af(0, T) + \partial_T f(0, T) = \theta(T) - \frac{\sigma^2}{2a}(1 - e^{-2aT}), \]

which is equivalent to (6).

A convenient representation. Looking at (6), it seems at first that we must use the differentiated term structure \(\partial_T f(0, T)\) to calibrate the model. That would be unfortunate, because differentiation amplifies the effect of observation-error. Actually, we can make do with \(f\) alone. Indeed, let’s look for a representation of the form

\[ r(t) = \alpha(t) + x(t) \tag{7} \]

where \(\alpha(t)\) is deterministic and \(x(t)\) solves

\[ dx = -ax dt + \sigma dw \quad \text{with} \quad x(0) = 0. \]

A brief calculation reveals that if \(\alpha' + a\alpha = \theta \quad \text{and} \quad \alpha(0) = r_0\)

then \(\alpha(t) + x(t)\) solves the SDE for \(r(t)\), and has the right initial condition, so (by uniqueness) it equals \(r(t)\). The ODE for \(\alpha\) is easy to solve: we have \((e^{at}\alpha)' = e^{at}\theta\), so

\[ \alpha(t) = r_0 e^{-at} + \int_0^t e^{-a(t-s)}\theta(s) ds. \]

Substituting (6) on the right, we get

\[ \alpha(t) = r_0 e^{-at} + \int_0^t \partial_s[e^{-a(t-s)}f(0, s)] + \frac{\sigma^2}{2a}e^{-a(t-s)}(1 - e^{-2as}) ds. \]

This simplifies to

\[ \alpha(t) = f(0, t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2. \]

Thus the decomposition (7) expresses \(r\) as the sum of two terms: a deterministic \(\alpha(t)\) reflecting the term structure at time 0, and a random process \(x(t)\) that’s entirely independent of market data.

Validity of Black’s formula. The situation is exactly the same as for Vasicek. The SDE for the interest rate under the forward-risk-neutral measure is

\[ dr = [\theta(t) - ar - \sigma^2 B(t, T)] dt + \sigma d\bar{w} \]

where \(d\bar{w}\) is a Brownian motion under this measure. This is simply a version of Hull-White with a different choice of \(\theta\). So bond prices are lognormal and Black’s formula is valid.

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The trinomial tree version of Hull-White. This topic is discussed quite clearly in Hull’s book Options, Futures, and other Derivatives (5th edition), sections 23.11 and 23.12. Please read it there.