Continuous Time Finance, Spring 2004 – Homework 4  
Posted 3/19/04, due 3/31/04

(1) To solve Problem 5 of HW3 you needed to know that if \( dr = (\theta - \alpha r) dt + \sigma dw \) then the function \( v(x, t) \) defined by

\[
v(x, t) = E_r(t=0) \left[ e^{-\int_0^t r(s) ds} f(r(T)) \right]
\]

solves

\[
v_t + (\theta - \alpha x)v_x + \frac{1}{2}\sigma^2 v_{xx} - xv = 0
\]

for \( t < T \), with final-time condition \( v(x, T) = f(x) \). This is a special case of the Feynman-Kac formula. Give a self-contained proof, using the method of HW1, problem 1. (You should assume that the PDE has a unique solution with this final-time condition; your task is to prove that the solution of the PDE satisfies (1).)

(2) The Section 6 notes explain how a trinomial tree can be used to approximate the random walk \( dx = \sigma dw \), and how working backward in this tree amounts to a standard finite-difference scheme for solving the backward Kolmogorov equation \( u_t + \frac{1}{2}\sigma^2 u_{xx} = 0 \). Let’s try to do something similar for the “geometric Brownian motion with drift” process \( dy = \mu y dt + \sigma y dw \), whose backward Kolmogorov equation is

\[
v_t + \mu y v_y + \frac{1}{2}\sigma^2 y^2 v_{yy} = 0
\]

Assume the time interval is \( \Delta t \), and at time \( t = n\Delta t \) the tree has nodes at \( -n\Delta y, \ldots, n\Delta y \). The process on the tree goes from \((y, t)\) to \((y + \Delta y, t + \Delta t)\) with probability \( p_u \), to \((y, t + \Delta t)\) with probability \( p_m \), and to \((y - \Delta y, t + \Delta t)\) with probability \( p_d \).

(a) How must \( p_u \), \( p_m \), and \( p_d \) be chosen to get the means and variances right? What are the conditions for them to be positive?

(b) What is wrong with this scheme?

(3) A better trinomial approximation of “geometric brownian motion with drift” is obtained by recognizing that if \( dy = \mu y dt + \sigma y dw \) then \( y = e^z \) with \( dz = (\mu - \frac{1}{2}\sigma^2) dt + \sigma dw \).

(a) Consider a trinomial tree process which goes from \((z, t)\) to \((z + \Delta z, t + \Delta t)\) with probability \( p_u \), to \((z, t + \Delta t)\) with probability \( p_m \), and \((z - \Delta z, t + \Delta t)\) with probability \( p_d \). How must \( p_u \), \( p_m \), and \( p_d \) be chosen to match the means and variances of the \( z \) process? What are the conditions for them to be positive?

(b) Working backward in this tree amounts to a finite-difference scheme for solving the backward Kolmogorov PDE \( w_t + (\mu - \frac{1}{2}\sigma^2) w_z + \frac{1}{2}\sigma^2 w_{zz} \) with specified final-time data at \( t = T \). In what sense can this also be viewed as a scheme for solving the PDE \( v_t + \mu y v_y + \frac{1}{2}\sigma^2 y^2 v_{yy} = 0 \)?

(Note: The “trinomial tree” scheme for valuing options uses this tree for the \( z \) process, with \( \mu = r \). However the option value is the discounted payoff; this introduces a discount factor of \( e^{-r\Delta t} \) at each timestep, and a term \(-rw\) in the PDE.)
(4) As we discussed in class, the general one-factor HJM model stipulates
\[ d_t f = \alpha(t, T) \, dt + \sigma(t, T) \, dw \] (2)
in the risk-neutral measure. We may choose the volatility \( \sigma(t, T) \) arbitrarily, but it determines the drift \( \alpha(t, T) \) through the formula
\[ \alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, u) \, du. \] (3)
The associated short rate is
\[ r(t) = f(0, t) + \int_0^t \sigma(s, t) \, dw(s) + \int_0^t \alpha(s, t) \, ds \]
which solves the SDE
\[ dr = \left[ \partial_T f(0, t) + \int_0^t \partial_T \sigma(s, t) \, dw(s) + \alpha(t, t) + \int_0^t \partial_T \alpha(s, t) \, ds \right] dt + \sigma(t, t) \, dw(t). \] (4)
Let’s verify that when \( \sigma(t, T) = \sigma e^{-a(T-t)} \) (with \( \sigma \) constant) we recover the Hull-White model:
(a) Show that \( \alpha(t, T) = \frac{\sigma^2}{a} e^{-a(T-t)} \left( 1 - e^{-a(T-t)} \right) \).
(b) Show that the SDE (4) reduces in this case to \( dr = (\theta(t) - ar) \, dt + \sigma \, dw \) with
\[ \theta(t) = \partial_T f(0, t) + af(0, t) + \frac{\sigma^2}{2a} \left( 1 - e^{-2at} \right). \]

(5) This problem revisits HW3, problem 1, using the general one-factor HJM theory \( d_t f(t, T) = \alpha(t, T) \, dt + \sigma(t, T) \, dw \) rather than Vasicek-Hull-White. Well, not the most general theory: you must assume for this problem that \( \sigma(t, T) \) is a given, deterministic function of \( t \) and \( T \) (whereas the general HJM framework permits it to be random, provided it depends only on time-\( t \) information). Besides the formulas (2)-(3), you’ll need the fact that
\[ d_t [P(t, T)/B_t] = [P(t, T)/B_t] \Sigma(t, T) \, dw \] (5)
where \( B_t \) is the money-market account and
\[ \Sigma(t, T) = -\int_t^T \sigma(t, u) \, du. \]
(a) Show that for \( t \leq \tau \leq T \leq S \), the random variable \( \ln[P(\tau, S)/P(\tau, T)] \) is normal under the risk-neutral measure, and its variance (given information at time \( t \)) is
\[ \int_t^\tau (\Sigma(u, S) - \Sigma(u, T))^2 \, du. \]
(b) To apply Black’s formula, we need the statistics of \( \ln[P(\tau, S)/P(\tau, T)] \) under the forward measure, not the risk-neutral measure. (The forward measure is the one for which \( V_t / P(t, T) \) is a martingale whenever \( V_t \) is the value of a tradeable.) Show that if \( \bar{w} \) is Brownian motion under the risk-neutral measure and \( \bar{w} \) is Brownian motion under the forward measure then

\[
d\bar{w} = -\Sigma(t, T) dt + dw.
\]

(Hint: specialize the calculation on page 9 of the Section 4 notes to the case at hand.)

(c) Use the result of (b) to show that \( \ln[P(\tau, S)/P(\tau, T)] \) is also normal under the forward measure, and its variance is the same under the forward and risk-neutral measures.

(d) Consider a call option with maturity \( T \) and strike \( K \), on a zero-coupon bond with maturity \( S > T \). Its payoff at time \( T \) is \( (P(T, S) - K)^+ \). Show that its value at time \( t \) is

\[
P(t, S)N(d_1) - KP(t, T)N(d_2)
\]

where

\[
d_1 = \frac{\ln[P(t, S)/P(t, T)K] + \frac{1}{2}s^2}{s}, \quad d_2 = d_1 - s
\]

where \( s \) is defined by

\[
s^2 = \int_t^T (\Sigma(u, S) - \Sigma(u, T))^2 du.
\]

(6) This problem revisits HW3, problem 2, using the general one-factor HJM theory. Consider the call option valued in problem 5.

(a) What trading strategy produces a replicating portfolio using tradeables \( P(t, S) \) and \( P(t, T) \)?

(b) What trading strategy produces a replicating portfolio using tradeables \( P(t, S) \) and the money market fund \( B_t \)?

(c) What trading strategy produces a replicating portfolio using two bonds \( P(t, T_1) \) and \( P(t, T_2) \), where \( T_1 \) and \( T_2 \) are arbitrary (distinct) values greater than \( T \)?