(1) Assume the Vasicek model $dr = (\theta - ar)dt + \sigma dw$ for the risk-neutral short rate process. Consider a call option with maturity $T$ and strike $K$, on a zero-coupon bond with maturity $S > T$. Its payoff at time $T$ is $(P(T, S) - K)_+$. Show using Black’s formula that the value of this option at time $t$ is

$$P(t, S)N(d_1) - KP(t, T)N(d_2)$$

where

$$d_1 = \frac{1}{\sigma_p} \log \frac{P(t, S)}{P(t, T)K} + \frac{1}{2} \sigma_p, \quad d_2 = \frac{1}{\sigma_p} \log \frac{P(t, S)}{P(t, T)K} - \frac{1}{2} \sigma_p$$

with

$$\sigma_p = \sigma \left( \frac{1 - e^{-2a(T-t)}}{2a} \right)^{1/2} B(T, S).$$

(The function $B(T, S)$ is the one from the representation $P(t, T) = A(t, T)e^{-B(t,T)r(t)}$.)

(2) Since Vasicek is a one-factor model, the call option of Problem 1 can be replicated by a self-financing trading strategy using any pair of tradeables.

(a) What trading strategy produces a replicating portfolio using tradeables $P(t, T)$ and $P(t, S)$?

(b) What trading strategy produces a replicating portfolio using tradeables $P(t, S)$ and the money market fund $B(t)$?

(c) What trading strategy produces a replicating portfolio using two bonds $P(t, T_1)$ and $P(t, T_2)$, where $T_1$ and $T_2$ are arbitrary (distinct) times greater than $T$?

(3) The call option of Problem 1 can be viewed as an exchange option involving the zero-coupon bonds $S_1 = KP(t, T)$ and $S_2 = P(t, S)$; indeed, its payoff at time $T$ is $(S_2 - S_1)_+$. We discussed exchange options in Section 3. Can the method we applied to exchange options be used to solve Problem 1? Explain.

(4) Suppose instead of Vasicek we use Hull-White for the short-rate: $dr = (\theta - ar)dt + \sigma dw$.

(a) Consider a call on a zero-coupon bond, as in Problem 1. Show that the valuation formula given in Problem 1 remains valid.

(b) Does this mean that the value of the option doesn’t depend on $\theta(t)$? Explain.

(5) Suppose we didn’t know the answer to Problem 1, and we decided to value the call by solving a PDE instead of by using Black’s formula. What PDE would we have to solve? (Be sure to specify the final-time condition). How does the solution determine the value of the option? (Note that by solving Problem 1, we have given a formula for the solution of this PDE!)