Calculus III Final Examination

Answer all questions and provide as much working as possible so that partial credit can be given if needed. Note: Final grades will be determined by scaling the exam and assignment grades and not by raw scores.

1. Consider the function

\[ f(x, y) = x^2y + y^3 - xy^2 - 9y \]

a) Calculate the partial derivatives of \( f(x, y) \) with respect to \( x \) and \( y \) and locate all the critical points. (4 marks)

b) Calculate the second partial derivatives of \( f \) and the discriminant

\[ D = f_{xx}f_{yy} - (f_{xy})^2. \]

Determine the nature of the critical points from a) and evaluate the local minima and maxima of \( f \) if appropriate. (5 marks)

c) Consider the square region

\[ \{(x, y) : |x| + |y| \leq 4\} \]

Determine the absolute maxima and minima of \( f \) in this region and their location. Show all your working and sketch your results (including the above region). (6 marks)

d) Using the information gathered above and/or calculated from level curves, sketch the function \( f \) on the \((x, y)\) plane. (6 marks)

e) Calculate the gradient vector field for \( f \). Find an equation for the tangent plane to the graph of \( z = f(x, y) \) at

\[ (x, y) = (0, 1). \]

Calculate a normal vector to this plane. (5 marks)

f) Suppose that we add the constraint that

\[ xy = 9 \]

(i.e. solutions must lie on this hyperbola). Find the maxima and/or minima of \( f \) subject to this constraint. Ignore the square region above in finding solutions. Hint: Use Lagrange multipliers. (4 marks)

2. Consider the three dimensional vector field \( \vec{F} = \left( \frac{yz}{x^2 + y^2 + z^2}, \frac{zx}{x^2 + y^2 + z^2}, z \right) \).

Evaluate exactly the surface integral

\[ \int_S \vec{F} \cdot dA \]

where \( S \) is the surface defined by the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \). Hint: Use a theorem do not try to do from first principles. (10 marks)
3. The moment of inertia of a rotating solid of uniform density $\rho$ in three dimensions occupying the region/solid $R$ rotating about the $z$ axis is given

$$I = \int \int \int_{R} (x^2 + y^2)\rho dV$$

Using an appropriate change of variables evaluate the moment of inertia of a solid which lies outside a cylinder of radius $a$ but inside a sphere of radius $2a$ (See the Figure below which shows a section of the top half of the solid in the $x - z$ plane). Show the derivation in detail. (20 marks)

4. Suppose that a force on an object is directed towards the origin and has magnitude $F(r)$ where $r$ is the distance from the origin.

a) Write down the force field in vector form for three dimensions. (8 marks)

b) Show that this is a conservative force field i.e. that it is the gradient of a scalar field. (4 marks)

c) Compute the total energy of the object and solve for the velocity. (4 marks)

d) Consider a portion of the object’s orbit/trajectory where its distance from the origin is decreasing with time. Show that work is done on the object by the force field as it travels along this segment. What happens if it travels in a circle? (4 marks)

5. Consider the three dimensional vector field

$$\vec{G} = (z, x, y)$$
and the elliptical paraboloidal surface $S$

$$z = x^2 + 3y^2$$

for

$$z \leq 1.$$

**a)** What is the boundary $C$ for the surface $S$? Write down a one parameter vector description of this boundary. Hint: Use the parametrization of an ellipse derived in class. (8 marks).

**b)** Evaluate the surface integral

$$\iint_S \text{curl} \vec{G} \cdot d\vec{S}$$

where $S$ is defined above. (6 marks)

**c)** Evaluate the line integral

$$\oint_C \vec{G} \cdot d\vec{r}$$

for the boundary curve $C$ for the surface $S$ derived in a). Confirm that Stokes theorem therefore holds for this case. (6 marks).