Mini Putnam Exam I

These are all taken from previous Putnam Exams.

1. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt{n} - \sqrt{m}, (n, m = 0, 1, 2, \ldots)$? Justify your answer.

2. Let $A$ and $B$ be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

3. Suppose $f$ and $g$ are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for every pair of real numbers $x$ and $y$,

$$f(x + y) = f(x)f(y) - g(x)g(y)$$
$$g(x + y) = f(x)g(y) + g(x)f(y)$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all $x$.

4. Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A perfect square is the square of an integer; that is a member of the set $\{0, 1, 4, 9, 16, \ldots\}$.) $a$ is within $n$ of $b$ if $b - n \leq a \leq b + n$.

5. What is the units (i.e., the rightmost) digit of $\left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor$? Here $[x]$ is the greatest integer $\leq x$.

6. Let $r$ and $s$ be positive integers. Derive a formula for the number of ordered quadruples $(a, b, c, d)$ of positive integers such that

$$3^r \cdot 5^s = \text{lcm}[a, b, c] = \text{lcm}[a, b, d] = \text{lcm}[a, c, d] = \text{lcm}[b, c, d]$$

The answer should be a function of $r$ and $s$. Note that lcm$[x, y, z]$ denotes the least common multiple of $x, y, z$.

7. For which real numbers $c$ is $(e^x + e^{-x})/2 \leq e^{cx^2}$ for all real $x$?

8. For which real numbers $a$ does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in the simplest form.)

9. Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining fours sides of length 2 units. Give the answer in the form $r + s\sqrt{t}$ with $r, s,$ and $t$ positive integers.