Some Putnam problems.

Here is a mini-exam for you, based on previous Putnam problems. Give yourself a solid 3 uninterrupted hours - no more - and do what you can.

1. For what region of the real \((a, b)\) plane, do both (possibly complex) roots of the polynomial \(z^2 + az + b = 0\) satisfy \(|z| < 1|\)?

2. Define \(f_0(x) = e^x\), \(f_{n+1}(x) = xf_n'(x)\). Show that \(\sum_{n=0}^{\infty} f_n(1)/n! = e^e\).

3. How many real roots does \(2^x = 1 + x^2\) have?

4. The real and imaginary parts of \(z\) are rational, and \(z\) has unit modulus. Show that \(|z^{2n} - 1|\) is rational for any integer \(n\).

5. Show that we cannot have 4 binomial coefficients

\[
\binom{n}{m}, \binom{n}{m+1}, \binom{n}{m+2}, \binom{n}{m+3}
\]

with \(n, m > 0\) (and \(m + 3 \leq n\)) in arithmetic progression.

6. Let \(\sum_{n=0}^{\infty} x^n(x-1)^{2n}/n! = \sum_{n=0}^{\infty} a_nx^n\). Show that no three consecutive \(a_n\) are zero.

7. Find all possible polynomials \(f(x)\) such that \(f(0) = 0\) and \(f(x^2 + 1) = f(x)^2 + 1\).

8. \(S\) is a set with a binary operation \(*\) such that (1) \(a * a = a\) for all \(a \in S\), and (2) \((a * b) * c = (b * c) * a\) for all \(a, b, c \in S\). Show that \(*\) is associative and commutative.