Putnam Exam: Integration problems

1987B1. Evaluate
\[ \int_{2}^{4} \frac{\sqrt{\ln(9 - x)} \, dx}{\sqrt{\ln(9 - x) + \ln(x + 3)}}. \]

1989A2 Evaluate \( \int_{0}^{a} \int_{0}^{b} e^{\max(b^2x^2, a^2y^2)} \, dy \, dx \), where \( a \) and \( b \) are positive.

1989B3. Let \( f \) be a function on \( [0, \infty) \), differentiable and satisfying
\[ f'(x) = -3f(x) + 6f(2x) \]
for \( x > 0 \). Assume that \( |f(x)| \leq e^{-\sqrt{x}} \) for \( x \geq 0 \) (so that \( f(x) \) tends rapidly to 0 as \( x \) increases). For \( n \) a nonnegative integer, define
\[ \mu_n = \int_{0}^{\infty} x^n f(x) \, dx \]
(sometimes called the \( n \)th moment of \( f \)).

a. Express \( \mu_n \) in terms of \( \mu_0 \).

b. Prove that the sequence \( \{\mu_n 3^n / n!\} \) always converges, and that the limit is 0 only if \( \mu_0 = 0 \).

1990B1. Find all real-valued continuously differentiable functions \( f \) on the real line such that for all \( x \)
\[ (f(x))^2 = \int_{0}^{x} ((f(t))^2 + (f'(t))^2 \, dt + 1990 \]

1991 A5. Find the maximum value of
\[ \int_{0}^{y} \sqrt{x^4 + (y - y^2)^2} \, dx \]
for \( 0 \leq y \leq 1 \).

1992A2. Define \( C(\alpha) \) to be the coefficient of \( x^{1992} \) in the power series expansion about \( x = 0 \) of \( (1 + x)^\alpha \). Evaluate
\[ \int_{0}^{1} C(-y - 1) \left( \frac{1}{y + 1} + \frac{1}{y + 2} + \frac{1}{y + 3} + \cdots + \frac{1}{y + 1992} \right) \, dy \]
1993A5. Show that
\[ \int_{-10}^{10} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{11}^{101} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{10}^{101} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx \]
is a rational number.

1993B4. The function \( K(x, y) \) is positive and continuous for \( 0 \leq x \leq 1, 1 \leq y \leq 1 \), and the functions \( f(x) \) and \( g(x) \) are positive and continuous for \( 0 \leq x \leq 1 \). Suppose that for all \( x, 0 \leq x \leq 1, \)
\[ \int_0^1 f(y)K(x, y)dy = g(x) \quad \text{and} \quad \int_0^1 g(y)K(x, y)dy = f(x). \]
Show that \( f(x) = g(x) \) for \( 0 \leq x \leq 1 \).

1995A2. For what pairs of \( (a, b) \) of positive real numbers does the improper integral
\[ \int_0^\infty \left( \sqrt[4]{x + a} - \sqrt[4]{x - b} \right) dx \]
converge?

1997A3. Evaluate
\[ \int_0^\infty \left( x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left( 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) dx. \]

2000A4. Show that the improper integral
\[ \lim_{B \to \infty} \int_0^B \sin(x) \sin(x^2) dx \]
converges.