Putnam Exam: Sequence problems

1985A3. Let $d$ be a real number. For each integer $m \geq 0$ define a sequence $\{a_m(j)\}$; $j = 0, 1, 2, \ldots$ by the condition
\[ a_m(0) = d/2^m, \quad \text{and} \quad a_m(j + 1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0 \]
Evaluate $\lim_{n \to \infty} a_n(n)$. 

1985B2. Define polynomials $f_n(x)$ for $n \geq 0$ by
\[ f_0(x) = 1, \quad f_n(0) = 0 \quad \text{for} \quad n \geq 1, \quad \text{and} \]
\[ \frac{d}{dx}(f_{n+1}(x)) = (n + 1)f_n(x + 1) \]
for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

1987B4. Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n \to \infty} x_n$ and $\lim_{n \to \infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

1990A1. Let
\[ T_0 = 2, \quad T_1 = 3, \quad T_2 = 6, \]
and for $n \geq 3$,
\[ T_n = (n + 4)T_{n-1} - 4nT_{n-2} + (4n - 8)T_{n-3}. \]
The first few terms are
\[ 2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392. \]
Find, with proof, a formula for $T_n$ of the form $T_n = A_n + B_n$, where $(A_n)$ and $(B_n)$ are well-known sequences.

1992A5. For each positive integer $n$, let
\[ a_n = \begin{cases} 0 & \text{if the number of 1's in the binary representation of } n \text{ is even}, \\ 1 & \text{if the number of 1's in the binary representation of } n \text{ is odd}. \end{cases} \]
Show that there do not exist positive integers $k$ and $m$ such that
\[ a_{k+j} = a_{k+m+j} = a_{k+2m+j}, \]
for $0 \leq j \leq m - 1$.

1992B3. For any pair $(x, y)$ of real numbers, a sequence $(a_n(x, y))_{n \geq 0}$ is defined as follows:
\[ a_0(x, y) = x, \]
\[ a_{n+1}(x, y) = \frac{a_n(x, y)^2 + y^2}{2} \quad \text{for} \quad n \geq 0. \]
Find the area of the region \( \{(x, y) | (a_n(x, y))_{n \geq 0} \text{ converges}\} \).

1993A2. Let \((x_n)_{n \geq 0}\) be a sequence of non-zero numbers such that
\[
x_n^2 - x_{n-1}x_{n+1} = 1 \quad \text{for } n = 1, 2, 3, \ldots.
\]
Prove there exists a real number \(a\) such that \(x_{n+1} = ax_n - x_{n-1}\) for all \(n \geq 1\).

1997A6. For a positive integer \(n\) and any real number \(c\), define \(x_k\) recursively by \(x_0 = 0\), \(x_1 = 1\), and for \(k \geq 0\),
\[
x_{k+2} = \frac{cx_{k+1} - (n - k)x_k}{k + 1}.
\]
Fix \(n\) and then take \(c\) to be the largest value for which \(x_{n+1} = 0\). Find \(x_n\) in terms of \(n\) and \(k\), \(1 \leq k \leq n\).

1999A6. The sequence \((a_n)_{n \geq 1}\) is defined by \(a_1 = 1, a_2 = 2, a_3 = 24,\) and, for \(n \geq 4,\)
\[
a_n = \frac{6a_{n-1}^2a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.
\]
Show that, for all \(n\), \(a_n\) is an integer multiple of \(n\).

2001B6. Assume that \((a_n)_{n \geq 1}\) is an increasing sequence of positive real numbers such that
\[\lim a_n/n = 0.\]
Must there exist infinitely many positive integers \(n\) such that \(a_{n-i} + a_{n+i} < 2a_n\) for \(i = 1, 2, \ldots, n - 1\)?

2002A5. Define a sequence by \(a_0 = 1\), together with the rules \(a_{2n+1} = a_n\) and \(a_{2n+2} = a_n + a_{n+1}\) for each integer \(n \geq 0\). Prove that every positive rational number appears in the set
\[
\left\{ \frac{a_{n+1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \ldots \right\}.
\]

2003B2. Let \(n\) be a positive integer. Starting with the sequence \(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\), form a new sequence of \(n - 1\) entries \(\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}\), by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of \(n - 2\) entries and continue until the final sequence produced as a single number \(x_n\). Show that \(x_n < \frac{2}{n}\).