(1) The interest rate in country XYZ is 36% per year. It has markets in which a commodity, the “widget”, can be traded spot and 6 months forward. The spot price of a widget is $1. (The currency in country XYZ is the US dollar.)

(a) What is the fair 6-month forward price for a contract for 1000 widgets? (Ignore the convenience value of holding widgets, and the cost of storing widgets.)

(b) You have access to financing in the US at an annual interest rate of 9% for up to $1,000,000.00. Assuming that the 6-month forward price for widgets is your answer to (a), construct an arbitrage. How much can you gain from this arbitrage?

(2) The current spot exchange rate between US dollars and Deutsche marks is 0.6676 $/DM. The price of a domestic 180-day Treasury bill is $98.0199 per $100 face value. The price of the equivalent German instrument is DM96.4635 per DM100 face value. The 180-day forward exchange rate is 0.66 $/DM. As usual we ignore transaction costs.

(a) What is the theoretical forward exchange rate?

(b) Since the current forward exchange rate is greater than the theoretical forward exchange rate, an arbitrage is possible with zero initial wealth. Describe such a strategy. How much does it gain, for a contract size of 100 DM?

(3) Ninety days ago you entered into a forward contract to buy a certain stock. The forward contract had a maturity of 100 days, and a delivery price of $50.25. Due to a change of circumstances you no longer want the stock. To offset it, you enter into a new forward contract to sell the stock. Assume the current stock price is $45, the interest rate is 4.75% per year, and the stock pays no dividend.

(a) What is the forward price for this new contract?

(b) What is the value of your net position when the two forward contracts mature?

(c) What is the present value, today, of your net position?

(4) Consider the following portfolio:

- long 1 call, strike price 40
- short 1 call, strike price 50
- short 1 call, strike price 70
- long 1 call, strike price 80.

All options are written on the same non-dividend-paying stock, with the same maturity date. Describe the value of the portfolio at maturity, as a function of the stock price \( S_T \), by giving its formula and sketching its graph. (Ignore the initial cost.)
Consider a portfolio containing two European call options: long a call with strike price 50 and short a call with strike price 55. Each option has maturity T.

(a) Describe the value of the portfolio at maturity, as a function of the stock price \( S_T \). (Ignore the initial cost.)

(b) Show that the prices of the two call options must satisfy

\[
0 \leq c[S_0, T, 50] - c[S_0, T, 55] \leq 5e^{-rT}
\]

where \( r \) is the interest rate.

Now consider a portfolio containing two European put options: short a put with strike price 50 and long a put with strike price 45. Each option has maturity T.

(c) Describe the value of the portfolio at maturity, as a function of the stock price \( S_T \). (Ignore the initial cost.)

(d) Show that the prices of the two put options must satisfy

\[
-5e^{-rT} \leq p[S_0, T, 45] - p[S_0, T, 50] \leq 0
\]

where \( r \) is the interest rate.

Finally consider the following portfolio: long one share of stock, short a call with strike price 55, long a put with strike price 45, and short a Treasury bill paying $50 at maturity. What is the relationship of this portfolio to those considered above?

(6) Which functions \( \phi(S_T) \) can be the value-at-maturity of a portfolio of calls? Such a portfolio consisting of \( a_i \) call options with strike price \( K_i \), \( 1 \leq i \leq N \), all having the same maturity date \( T \). (We permit short as well as long positions, i.e. \( a_i \) can be positive or negative. We may suppose \( 0 < K_1 < \ldots < K_N \). The value of this portfolio at maturity is \( \phi(S_T) = \sum_{i=1}^{N} a_i(S_T - K_i)_+ \).)

(a) Show that \( \phi \) is a continuous, piecewise linear function of \( S_T \), with \( \phi(S_T) = 0 \) for \( S_T \) near 0, and \( \phi(S_T) = a_\infty S_T + b_\infty \) when \( S_T \) is sufficiently large.

(b) Show that any such \( \phi \) can be realized by a suitable portfolio, and the portfolio is uniquely determined by \( \phi \). (Hint: relate the second derivative \( \phi'' \) to the parameters \( a_i \) and \( K_i \).)

(c) Show that \( a_\infty = \sum_{i=1}^{N} a_i \) and \( b_\infty = -\sum_{i=1}^{N} a_i K_i \).