Assignment 6, due November 7

Corrections: (none yet)

See notes on the Ito Integral and diffusions posted on the Resources page.

1. Consider the incorrect approximation formula for $\int_0^t W_s \, dW_s$:

$$Y_{\Delta t}^\Delta t = \sum_{t_k < t} W_{t_{k+1}} (W_{t_{k+1}} - W_{t_k}) \ .$$ (1)

Repeat (changing as necessary) the steps that lead to

$$X_{\Delta t}^\Delta t \rightarrow X_t = \frac{1}{2} (W_t^2 - t)$$

to find a formula for

$$Y_t = \lim_{\Delta t \rightarrow 0} Y_{\Delta t}^\Delta t \ .$$

Show that $Y_t \neq X_t$. Explain the difference by calculating

$$E\left[W_{tk+1} (W_{tk+1} - W_{tk})\right] \neq E\left[W_{tk} (W_{tk+1} - W_{tk})\right] \ .$$

This calculation predicts a formula for $Y_t - X_t$. Find it.

2. The urn process (or one of the processes called urn process) has a number of red and blue balls totaling $n$. The number of red balls at time $t_k$ is $N_k$. At each time, you remove one of the balls (chosen independently and at random, with each ball equally likely to be chosen) and replace it with a new independent ball with probability $p$ to be blue and $1 - p$ to be red.

(a) Calculate the probabilities for $N \rightarrow N + 1$ and $N \rightarrow N - 1$ (and therefore $N \rightarrow N$) in one step of the process. Use this to calculate $E[\Delta N \mid \mathcal{F}_k]$ and $E[(\Delta N)^2 \mid \mathcal{F}_k]$.

(b) Write an SDE for $N_t$ assuming $\Delta t = 1$ and $t_k = k\Delta t$. Take $dt = 1$ in this calculation. This calculation is sort of correct and sort of incorrect. We will refine it by using systematic scalings next week.

3. Consider a chemical system in which there are $N_t$ copies of a certain kind of molecule at time $t$. In a time $dt$, there is probability $\lambda dt$ that a new molecule of this type “arrives” (is produced by some chemical reaction). Molecules of this type are removed by combining with each other to form...
dimers (compound molecules consisting of two of the originals bound together). The probability that a given molecule can find a partner and form a dimer is $\mu(N_t - 1)dt$. Here $N_t - 1$ is the number of other molecules that this one could combine with. Write an approximate SDE for this process, assuming $N_t$ is large. The difference between $N_t$ and $N_t - 1$ should not matter if $N$ is large.

4. The *Ornstein Uhlenbeck* process is the one we saw a few weeks ago: $dV_t = -\gamma V_t dt + \sigma dW_t$. Suppose $X_0 = 0$ and $dX_t = V_t dt$. Show that for large $t$, $E[X_t^2] \approx Dt$ and find a formula for $D$ in terms of $\gamma$ and $\sigma$. Hint: express $X_t$ as a double integral of $dW_t$ and reverse the order of integration. Use the previous expression for $V_t$ as the first step. This formula was first derived by Einstein in his model of Brownian motion.