Assignment 5, due October 24

Corrections: (none yet)

See notes on the Ito Integral posted on the Resources page.

1. Find a formula for \( \int_0^t (W_s^2 - s) \, dW_s \) in terms of \( W_t \). Hint: use Ito's lemma to calculate the differentials of \( W_t^3 \) and \( tW_t \).

2. The stochastic process \( X_t \) is geometric Brownian motion if it satisfies a stochastic differential equation (see the notes) of the form
   \[ dX_t = \mu X_t \, dt + \sigma X_t \, dW_t. \]
   Show that stochastic processes of the form
   \[ X_t = X_0 e^{\sigma W_t + (\mu - \frac{1}{2} \sigma^2) t} \]
   are geometric Brownian motions.

3. Suppose \( X_t = \int_0^t F_s \, dW_s + \int_0^t G_s \, ds \).
   (a) Show that the Ito integral is additive in the sense that \( \int_0^{t'} F_s \, dW_s = \int_0^t F_s \, dW_s + \int_t^{t'} F_s \, dW_s \). For this you need to define approximations that do not start at \( t = 0 \) and show that additivity holds approximately for the approximations.
   (b) If \( t' = t + \Delta t \) for small \( \Delta t \), you may use the approximations \( F_s \approx F_t \) and \( G_s \approx G_t \) for \( t \leq s \leq t' \). The increment of \( X \) over this interval is \( \Delta X = X_{t+\Delta t} - X_t \). Show that
      \[ E[\Delta X \mid F_t] \approx G_t \Delta t \]
      \[ E[(\Delta X)^2 \mid F_t] \approx F_t^2 \Delta t \]
      \[ \text{var}[\Delta X \mid F_t] \approx F_t^2 \Delta t \]
      The conditional variance in the last formula is the conditional expected value of the square difference from the conditional mean. These formulas may be written informally as \( E[dX_t \mid F_t] = G_t \, dt \) and \( E[(dX_t)^2 \mid F_t] = F_t^2 \, dt \).
   (c) (not an action item) The formulas of part (b) usually are used in reverse. In modeling we have estimates of \( E[\Delta X \mid F_t] \) and \( E[\Delta X^2 \mid F_t] \) and use them to discover \( F \) and \( G \), or the coefficients \( a(X) \) and \( b(X) \) in the stochastic differential equation modeling \( X \). For example, the geometric Brownian motion of a stock price is “derived” (“motivated might be more accurate) by saying \( E[(\Delta X/X_t) \mid F_t] = \mu \Delta t \) (rate of return is \( \Delta X/(X_t \Delta t) \)). This is constant expected rate of return = \( \mu \).) Similarly, \( \text{var}(\Delta X/X_t \mid F_t) = \sigma^2 \Delta t \) defines \( \sigma \) as the volatility.
4. The Ornstein Uhlenbeck process is the linear stochastic differential equation
\[ X_t = -\gamma X_t \, dt + \sigma dW_t. \] (1)

(a) Show that the solution is given by a formula something like (part of the problem is to find the correct formula)
\[ X_t = e^{-\gamma t} X_0 + \int_0^t e^{-\gamma (t-s)} dW_s. \] (2)

(b) Assuming that \( X_0 \) is not random, use the (corrected) formula (2) to find formulas for \( m_t = E[X_t] \) and \( s^2_t = E[X^2_t] \). Evaluate these in the limit \( t \to \infty \).

(c) As an alternative method, calculate \( dm_t \) by formally differentiating \( m = E[X] \), as \( dm_t = E[dX_t | \mathcal{F}_t] \). Show that you get the same result by using the first formula of part (3b) and taking the limit \( \Delta t \to 0 \).

(d) Find formula for \( ds_t \) in a similar way using the middle part of (3b). For this, write \( (X_t + \Delta X)^2 = X^2_t + 2X_t \Delta X + \Delta X^2 \) and take the conditional expectation in \( \mathcal{F}_t \).

(e) Intuition might suggest that the probability distribution of \( X_t \) would converge so a limiting distribution as \( t \to \infty \). The term \( -\gamma X_t \, dt \) acts as a restoring force, pushing \( X_t \) toward zero from the positive and negative directions. The restoring force becomes stronger as \( |X_t| \) increases but the noise keeps the same strength. If \( X_t \) were in equilibrium, we would have \( dm_t = 0 \) and \( ds_t = 0 \). Use the results of part (d) to find the values of \( m_t \) and \( s_t \) corresponding to this steady state. If everything is correct, these values should agree with those of part (b).