Assignment 8.

Given November 11, due November 18. Last revised, November 11.

**Objective:** Diffusions and diffusion equations.

1. An Ornstein Uhlenbeck process is a stochastic process that satisfies the stochastic differential equation

\[ dX(t) = -\gamma X(t)dt + \sigma dW(t). \]  

(a) Write the backward equation for \( f(x, t) = E_x[V(X(T))]. \)

(b) Show that the backward equation has (Gaussian) solutions of the form \( f(x, t) = A(t) \exp(-s(t)(x - \xi(t))^2/2). \) Find the differential equations for \( A, \xi, \) and \( s \) that make this work.

(c) Show that \( f(x, t) \) does not represent a probability distribution, possibly by showing that \( \int_{-\infty}^{\infty} f(x, t)dx \) is not a constant.

(d) What is the large time behavior of \( A(t) \) and \( s(t) \)? What does this say about the nature of an Ornstein Uhlenbeck reward that is paid long in the future as a function of starting position?

2. The forward equation:

(a) Write the forward equation for \( u(x, t) \) which is the probability density for \( X(t). \)

(b) Show that the forward equation has Gaussian solutions of the form

\[ u(x, t) = \frac{1}{\sqrt{2\pi} \sigma(t)} e^{-(x - \mu(t))^2/2\sigma^2(t)}. \]

Find the appropriate differential equations for \( \mu \) and \( \sigma. \)

(c) Use the explicit solution formula for (1) from assignment 7 to calculate \( \mu(t) = E[X(t)] \) and \( \sigma(t) = \text{var}[X(t)]. \) These should satisfy the equations you wrote for part b.

(d) Use the approximation from (1): \( \Delta X \approx -\gamma X \Delta t + \sigma \Delta W \) (and the independent increments property) to express \( \Delta \mu \) and \( \Delta(\sigma^2) \) in terms of \( \mu \) and \( \sigma \) and get yet another derivation of the answer in part b. Use the definitions of \( \mu \) and \( \sigma \) from part c.

(e) Differentiate \( \int_{-\infty}^{\infty} xu(x, t)dx \) with respect to \( t \) using the forward equation to find a formula for \( d\mu/dt. \) Find the formula for \( d\sigma/dt \) in a similar way from the forward equation.
f. Give an abstract argument that $X(t)$ should be a Gaussian random variable for each $t$ (something is a linear function of something), so that knowing $\mu(t)$ and $\sigma(t)$ determines $u(x,t)$.

g. Find the solutions corresponding to $\sigma(0) = 0$ and $\mu(0) = y$ and use them to get a formula for the transition probability density (Green’s function) $G(y,x,t)$. This is the probability density for $X(t)$ given that $X(0) = y$.

h. The transition density for Brownian motion is $G_B(y,x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right)$. Derive the transition density for the Ornstein Uhlenbeck process from this using the Cameron Martin Girsanov formula (warning: I have not been able to do this yet, but it must be easy since there is a simple formula for the answer. Check the bboard.).

i. Find the large time behavior of $\mu(t)$ and $\sigma(t)$. What does this say about the distribution of $X(t)$ for large $t$ as a function of the starting point?

3. Duality:

a. Show that the Green’s function from part 2 satisfies the backward equation as a function of $y$ and $t$.

b. Suppose the initial density is $u(x,0) = \delta(x-y)$ and that the reward is $V(x) = \delta(x-z)$. Use your expressions for the corresponding forward solution $u(x,t)$ and backward solution $f(x,t)$ to show by explicit integration that $\int_{-\infty}^{\infty} u(x,t)f(x,t)dx$ is independent of $t$. 