Assignment 5.

Given October 1, due October 21. Last revised, October 7.

Objective: Brownian Motion.

1. Suppose \( h(x) \) has \( h'(x) > 0 \) for all \( x \) so that there is at most one \( x \) for each \( y \) so that \( y = h(x) \). Consider the process \( Y_t = h(X_t) \), where \( X_t \) is standard Brownian motion. Suppose the function \( h(x) \) is smooth. The answers to the questions below depend at least on second derivatives of \( h \).

   a. With the notation \( \Delta Y_t = Y_{t+\Delta t} - Y_t \), for a positive \( \Delta t \), calculate \( a(y) \) and \( b(y) \) so that \( E[\Delta Y_t | F_t] = a(Y_t) \Delta t + O(\Delta t^2) \) and \( E[\Delta Y_t^2 | F_t] = b(Y_t) \Delta t + O(\Delta t^2) \).

   b. With the notation \( f(Y_t, t) = E[V(Y_T) | F_t] \), find the backward equation satisfied by \( f \). (Assume \( T > t \).)

   c. Writing \( u(y, t) \) for the probability density of \( Y_t \), use the duality argument to find the forward equation satisfied by \( u \).

   d. Write the forward and backward equations for the special case \( Y_t = e^{cX_t} \). Note (for those who know) the similarity of the backward equation to the Black Scholes partial differential equation.

2. Use a calculation similar to the one we used in class to show that \( Y_T = X_T^4 - 6 \int_0^T X_t^2 dt \) is a martingale. Here \( X_t \) is Brownian motion.

3. Show that \( Y_t = \cos(kX_t)e^{k^2 t/2} \) is a martingale.

   a. Verify this directly by first calculating (as in problem 1) that

   \[
   E[Y_{t+\Delta t} | F_t] = Y_t + O(\Delta t^2)
   \]

   Then explain why this implies that \( Y_t \) is a martingale exactly (Hint: To show that \( E[Y_t | F_t] = Y_t \), divide the time interval \((t, t')\) into \( n \) small pieces and let \( n \to \infty \).

   b. Verify that \( Y_t \) is a martingale using the fact that a certain function satisfies the backward equation. Note that, for any function \( V(x) \), \( Z_t = E[V(X_T) | F_t] \) is a martingale (the tower property). Functions like this \( Z \) satisfy backward equations.

   c. Find a simple intuition that allows a supposed martingale to grow exponentially in time.

4. Let \( A_{x_0, t} \) be the event that a standard Brownian motion starting at \( x_0 \) has \( X_{t'} > 0 \) for all \( t' \) between 0 and \( t \). Here are two ways to verify the large time asymptotic approximation

\[
P(A_{x_0, t}) \approx \frac{1}{\sqrt{2\pi \sqrt{t}}} e^{-\frac{x_0^2}{2t}}.
\]
a. Use the formula from “Images and reflections” to get
\[
P(A_{x_0,t}) = \int_0^\infty u(x,t) dx \\
\approx \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-x^2/2t} \left( e^{x_0/t} - e^{-x_0/t} \right) dx.
\]

The change of variables \(y = x/\sqrt{t}\) should make it clear how to approximate the last integral for large \(t\).

b. Use the same formula to get
\[
\frac{-d}{dt} P(A_{x_0,t}) = \frac{1}{\sqrt{2\pi}} \frac{2x_0}{t^{3/2}} e^{-x_0^2/2t}.
\]

Once we know that \(P(A_{x_0,t}) \to 0\) as \(t \to \infty\), we can estimate its value by integrating (1) from \(t\) to \(\infty\) using the approximation \(e^{\text{const}/t} \approx 1\) for large \(t\). Note: There are other hitting problems for which \(P(A_t)\) does not go to zero as \(t \to \infty\). This method would not work for them.