Assignment 7.

Given October 31, due November 7 Last revised November 5.

Objective: The Ito integral

1. We want to calculate from the definition

\[ Y_T = \int_0^T X_t^2 dX_t , \]

where \( X_t \) is standard Brownian motion. Were \( X_t \) a differentiable function of \( t \), the answer would be \( Y_T = \frac{1}{3} X_T^3 \). There will be an Ito correction to this. To find the correction, use the notations \( X_k = X_{t_k} \), \( t_k = k\Delta t \), and \( \Delta X_k = X_{k+1} - X_k \), then calculate

\[ X_{k+1}^3 - X_k^3 = 3X_k^2 \Delta X_k + 3X_k \Delta X_k^2 + \Delta X_k^3. \]

Turning this around gives

\[ X_k^2 \Delta X_k = \frac{1}{3}(X_{k+1}^3 - X_k^3) - X_k \Delta X_k^2 - \frac{1}{3} \Delta X_k^3. \]

Thus

\[ Y_T^{(n)} = \sum_{k=0}^{n-1} X_k^2 \Delta X_k = \frac{1}{3} X_T^3 - A_n - B_n - \frac{1}{3} C_n , \]

where

\[ A_n = \sum_{k=0}^{n-1} X_k \Delta t \ , \]
\[ B_n = \sum_{k=0}^{n-1} X_k (\Delta X_k^2 - \Delta t) \ , \]
\[ C_n = \sum_{k=0}^{n-1} \Delta X_k^3 \ . \]

Find the limit as \( \Delta t \to 0 \) of \( A_n \) as an integral involving \( X_t \). Show that \( B_n \to 0 \) and \( C_n \to 0 \) as \( n \to \infty \) with \( n = 2^L \) by calculating \( E[B_n^2] \) and \( E[C_n^2] \). Note that if \( \sum_n E[B_n^2] < \infty \) then \( \sum_n B_n^2 < \infty \) almost surely (why?), which implies that \( B_n^2 \to 0 \) as \( n \to \infty \) almost surely.

2. Use the Ito isometry to calculate \( E[Y_T^2] \), where

\[ Y_T = \int_0^T \left( \int_{s=0}^t (t-s)X_s dX_s \right) dX_t \ . \]

It may help to find a formula for \( E[Z^4] \) and \( E[Z^6] \) where \( Z \sim \mathcal{N}(0, \sigma^2) \).
3. We want simple approximations to the integral

\[ Y_T = \int_0^T V(X_t) dX_t \]

when \( T \) is small. For any random variable, \( W_T \), we say that \( W_T = O(t^p) \) if \( E[|W_T|] = O(\Delta t^p) \). Note that this is not the same as saying that there is a constant \( C \) so that \( |W_T| \leq C t^p \). For example, if \( X_T \) is standard Brownian motion, we already used the fact that \( X_T = O(\sqrt{T}) \).

a. Use the Cauchy Schwartz inequality to show that if \( W_T^2 = O(\Delta t^p) \) then \( W_T = O(\Delta t^p) \).

b. Use the Ito isometry formula to show that if \( F_t \in \mathcal{F}_t, F_t^2 = O(t^p) \), and \( W_T = \int_0^T F_t dX_t \), then \( W_t = O(t^{p+1}/2) \).

c. Suppose \( V(x) \) is a smooth function of \( x \) and take enough Taylor series terms of \( V \) to show that

\[ Y_T = V(X_0) \Delta X_T + \frac{1}{2} V'(X_0)(\Delta X_T^2 - T) + O(T^{3/2}) \, . \]

We suppose the Brownian motion path \( X_t \) starts at \( X_0 \), which is not necessarily zero. We use the notation \( \Delta X_T = X_T - X_0 \). Hint: integrate the Taylor series for \( V(X_t) \) term by term, and use parts a-c to get a bound for what we get by integrating the remainder term.

4. Find a backwards equation for

\[ f(x, t) = E_{x,t} \left[ \exp \left( \int_t^T V(X_s) dX_s \right) \right] \, . \]

Hint: First derive the formula using the tower property for conditional expectation,

\[ f(x, t) = E_{x,t} \left[ \exp \left( \int_{s=t}^{t+\Delta t} V(X_s) dX_s \right) \cdot f(x + \Delta X, t + \Delta t) \right] \, , \]

then use Taylor series, including the answer to question 3.