Assignment 2.

Given September 12, due September 19.
Last revised, September 15.

Objective: Conditioning and Markov chains.

1. Suppose that \( F \) and \( G \) are two algebras of sets and that \( G \) adds information to \( F \) in the sense that any \( F \) measurable event is also \( G \) measurable. Since \( F \) and \( G \) are collections of events, this may be written \( F \subseteq G \). Suppose that the probability space \( \Omega \) is finite and that \( X(\omega) \) is a variable defined on \( \Omega \) (that is, a function of the random variable \( \omega \)). The conditional expectations (in the modern sense) of \( X \) with respect to \( F \) and \( G \) are
   \[ Y = E[X \mid F] \] and \( Z = E[X \mid G] \). In each case below, state whether the statement is true or false and explain your answer with a proof or a counterexample.
   a. \( Z \in F \).
   b. \( Y \in G \).
   c. \( Z = E[Y \mid G] \).
   d. \( Y = E[Z \mid F] \).

2. For any event \( A \subseteq \Omega \) we can define the “indicator function” (also called the “characteristic function”, particularly by people who learned probability late in life), \( 1_A(\omega) = 1 \) if \( \omega \in A \) and \( 1_A(\omega) = 0 \) if \( \omega \notin A \). People who call this the “characteristic function” (usually people who learned probability late in life) write \( \chi_A(\omega) \) for \( 1_A(\omega) \).
   a. Show that \( E[1_A] = P(A) \).
   b. For any event, \( B \), show that the classical \( E[1_A \mid B] \) is the same as the Bayes’ rule conditional probability of \( A \).

The “modern” definition of conditional probability, conditioning on an algebra of sets rather than a single set, is \( P(A \mid F) = E[1_A \mid F] \). The relationship between the classical and modern definition of conditional probability is more or less the same as the relation between classical and modern expected value.

3. Let \( S \) be a finite state space and \( \Omega \) be the set of paths of length \( T \) from \( S \). Let \( P(X) \) be the probability of a path \( X \in \Omega \). For any \( t \) in the range \( 1 < t < T \), let \( F_t \) be the algebra of events in \( \Omega \) generated by the values of \( X_s \) for \( 1 \leq s \leq t \). Let \( G_t \) be the smaller algebra generated only by \( X_t \). Finally, let \( H_t \) be the “complementary” algebra (based on complementary information) generated by the values \( X_t, \ldots, X_T \). An event in \( H_t \) is a statement about the path from time \( t \) on without saying anything about the beginning values \( X_1, \ldots, X_{t-1} \). Show that the Markov property is equivalent to either of the following,
a. 
\[ E[1_A | \mathcal{F}_t] = E[1_A | \mathcal{G}_t] \quad \text{for any } A \in \mathcal{H}_{t+1}. \]

b. 
\[ E[F(X) | \mathcal{F}_t] = E[F(X) | \mathcal{G}_t] \quad \text{for any } F \in \mathcal{H}_{t+1}. \]

Notes: (i) Part a is a special case of part b (why?). (ii) Part b implies that \( E[F(X) | \mathcal{F}_t] \) is a function of \( X_t \) only. This is supposed to be intuitively clear as a consequence of the Markov property. (iii) Together with question 1, part b is a justification for the backward equation for expected values of final payouts.

4. Suppose we have a 3 state Markov chain with transition matrix
\[
P = \begin{pmatrix}
.6 & .2 & .2 \\
.3 & .5 & .2 \\
.1 & .2 & .7 \\
\end{pmatrix}
\]
and suppose that \( X_1 = 1 \).

a. Show that the probability distribution of the first \( t \) steps conditioned on \( \mathcal{G}_{t+1} \) is the same as that conditioned on \( \mathcal{H}_{t+1} \). This is a kind of backwards Markov property: a forward Markov chain is a backward Markov chain also.

b. Calculate \( P(X_2 = 2 | \mathcal{G}_3) \). This consists of 3 numbers.