Probabilists often use a different normalization for Hermite polynomials than the one in homework 3. Adopting this, we define $H_n(x)$ by

$$H_n(x) e^{-x^2/2} = \partial_x e^{-x^2/2}.$$  

(1)

It is common to put in a factor of $(-1)^n$ so that $H_n$ has the form $x^n + \cdots$. Our $H_3$ is $-x^3 + 3x$, not $x^3 - 3x$.

1. The exponential generating function (as opposed to the “ordinary” generating function) for the $H_n$ is

$$F(x, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x).$$

The ordinary generating function is missing the factor $1/n!$. Use the formula (1) to find $F(x, z)$ explicitly. Use this to find an integral formula for $H_n(x)$. This formula can be used to derive approximations to $H_n(x)$ when $n$ and $x$ are large.

2. With the notation $g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, show that

$$\int H_n(x) H_m(x) g(x) dx = 0 \quad \text{when } m \neq n,$$

and evaluate

$$\int H_n(x)^2 g(x) dx.$$

Hint: Show that $x^k e^{-x^2/2} \sim H_k(\partial_x) e^{-x^2/2}$ (e.g. using the Fourier transform). Suppose that $m < n$ and show that

$$H_m(x) \partial_x^n e^{-x^2/2} = \partial_x^{n-m} P(\partial_x) e^{-x^2/2},$$

where $P$ is some polynomial. For $m = n$ this is almost true.

3. Show that the Hermite polynomials are a (complete) basis for the Hilbert space $L_2^g$, which has inner product

$$\langle u, v \rangle_g = \int \overline{u}(x) v(x) g(x) dx.$$

Do do this, compute the Hermite polynomial expansion of $e^{i\xi x}$ (formula (1) will be helpful here) and show explicitly that it converges to $e^{i\xi x}$. Why is this enough?
4. Suppose that $X$ is a standard normal random variable and $f(x)$ has $E(f^2(X)) < \infty$. Interpret the Hermite expansion of $f$ as a representation of the random variable $f(X)$ as a linear combination of “Hermite” random variables $H_n(X)$.

5. Here is the extension to $n$ variables. A multi-index is a list of $n$ non negative integers: $\alpha = (\alpha_1, \ldots, \alpha_n)$. The notation $x^\alpha$ means $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\alpha_x = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$. The degree of a multi index is $p = |\alpha| = \alpha_1 + \cdots + \alpha_n$. The multidimensional Hermite polynomials are

$$H_{\alpha}(x) = H_{\alpha_1}(x_1) \cdots H_{\alpha_n}(x_n) = e^{\frac{|x|^2}{2}} \partial^{\alpha}_x e^{-|x|^2/2} . \quad (2)$$

Show that the Hilbert space $L^2_g(R^n)$ is a direct orthogonal sum of the degree $p$ subspaces

$$S_p = \text{span of } \{ H_{\alpha} \mid |\alpha| = p \} .$$

6. For any orthogonal $n \times n$ matrix, $Q$, and any function $f \in L^2_g(R^n)$, define $Qf$ by $(Qf)(x) = f(Qx)$. Clearly $\|Qf\|_g = \|f\|_g$.

a. Verify by direct calculation that the space $S_2$ is invariant under the action of $Q$ for $n = 2$. That is, show that if $x' = x \cos(\theta) + y \sin(\theta)$ and $y' = y \cos(\theta) - x \sin(\theta)$, then

$$H_2(x') = a(\theta)H_2(x) + b(\theta)H_1(x)H_1(y) + c(\theta)H_2(y) ,$$

and find a similar representation for $H_1(x')H_1(y')$.

b. Give a simpler proof that each $S_p$ is invariant for any $n$. Hint: what does $Q$ do to $\partial^\alpha_x$?

c. Compute $f_p(x, \xi) = S_p e^{i \xi \cdot x}$. Here, I have used $S$ to denote the orthogonal projection onto the space $S$. Hint: the right $Q$ can make this easy.

7. Now let $(\Omega, \mathcal{F}, \mu)$ represent Wiener measure for Brownian motion on the interval $[0, 1]$ with $W(0) = 0$. Let $\mathcal{F}_L \subset \mathcal{F}$ be the algebra of sets generated by the diadic interval differences

$$G_{L,k} = W((k + 1)2^{-L}) - W(k2^{-L}) , \quad \text{for } k = 0, 1, \ldots, 2^L - 1 .$$

For any $f(W)$ (I will use $W$ instead of $\omega$ to represent the basic random element of $\Omega$: an element of $\Omega$ is a path.) with $E[|f(W)|^2] < \infty$ define

$$f_{L,p}(W) = S_p E[f \mid \mathcal{F}_L] .$$

Here, we use $S_p$, as in part 6c, to be the orthogonal projection onto the space of order $p$ in the space of functions of $n = 2^L$ independent gaussians. Show that, for each $p$, $f_{L,p}$ is a martingale as a function of $L$. Show that the limit

$$\lim_{L \to \infty} f_{L,p}(W) = f_p(W)$$
exists, that $f(W) = \sum_{p=0}^{\infty} f_p(W)$, and that

$$E[f^2(W)] = \sum_{p=0}^{\infty} E[f_p^2(W)].$$

8. Show that for each $f$ and $p$ there is a function $a_p(t_1, \ldots, t_p)$ with

$$f_p(W) = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{p-1}} a_p(t_1, \ldots, t_p) dW(t_p) \cdots dW(t_1).$$

Hint: Take a limit of approximations $a_{L,p}(t_1, \ldots, t_p)$, which are martingales (in $L$) for each $p$ as functions of $(t_1, \ldots, t_p)$ (show this). Note that in the limit, the energy in all terms involving $H_2(x) = x^2 - 1$ or higher goes to zero. These spaces are the Wiener chaos spaces of degree $p$. 